A Lagrangian Approach to Modeling and Analysis of a Crowd Dynamics

Sudipto Mukherjee, Debdipta Goswami, and Sarthak Chatterjee

***Abstract*—Modeling of crowd and pedestrian dynamics has intrigued engineers, physicists, and sociologists alike, even in recent times. Scientists have long sought to model the collective motion of large groups of individuals and study the mathematical basis of what seems to be apparently random behavior. A large number of macroscopic models have been proposed that describe crowd motion as a whole, much like the partial differential equations of fluid mechanics. This paper proposes a Lagrangian approach to the modeling of crowd dynamics by taking into consideration the various forces that act between the members of a crowd while they are in motion in a 2-D field. We attempt a realistic modeling of the attractive and repulsive forces between the members and seek to give a definite mathematical backbone to the terms "panic" and "evacuation." That the dynamics is stable is demonstrated by constructing an appropriate Lyapunov energy function. We then linearize the dynamics to obtain mathematical expressions for the small perturbations about an equilibrium point. Through machine simulations and by tracking the motion of actual crowd systems, we show the validity of the mathematics of group formation and evacuation that we have proposed.**

***Index Terms*—Crowd dynamics, evacuation, macroscopic and microscopic model, modeling, panic.**

I. Introduction

CROWD dynamics remains to be an area of focus in multiagent systems even in [10] and [14]. Pedestrian and crowd dynamics seek to mimic the many complexities exhibited by a crowd in its immediate surroundings. The social behavior of the entity as a whole, as well as the individuals forming the group, has received due interest from researchers from various fields of study including physics, sociology, and engineering. Crowd dynamics may be defined as the study of the formation of crowds, the interactive forces governing the motion of individuals, and the effects at high density, with the motivation for building better, safer, and robust crowd management systems. While the literature is replete with works centered around swarming dynamics, social foraging swarms, automated multiagent systems, and heuristics that represent natural processes, the modeling of crowd behavior still has a lot of scope for future contributions [8]. Modeling and analysis of crowd dynamics has gained momentum recently owing to the need to devise effective evacuation procedures. Such evacuation processes are quintessential to the survival and safety of large numbers of people. Intuitive methods may provide the necessary evacuation arrangements in simple situations, but when the architectural structures are more complex and in situations of utmost exigency [25], [28], [29], [34] prudent evacuation decisions rest on reliable simulations.

Evacuation processes [46], crowd disasters [20], and traffic management may have been central to the motivation for research in this field. However, it would be grossly unjustified to undermine the interest that is culminated by the unique social collective behavior of a crowd. Even though each individual is distinct, there occurs an underlying coherence that governs crowd behavior [9], [12], [40], [44]. The growing interest of researchers to opt for a macroscopic model of crowd dynamics can be attributed to a crowd's underlying collective behavior. Crowd simulation models that have existed for a few decades include queuing models, transition matrix models, and stochastic models [7], [15], [52]. Earlier models resorted to statistical regression analysis for predictions. Henderson [23] modeled his crowd in analogy with the behavior of fluids. Although this model did portray collective behavior, it had to be corrected to include interactions introduced by individuals, which did not pertain to momentum conservation and energy conservation laws. Such fluid dynamic models were applied to traffic flow problems and 1-D vehicle flow problems.

It is apt now to refer to the noteworthy distinction that exists between two disparate methods of crowd modeling—the macroscopic and the microscopic methods. Microscopic modeling focuses on the individuals forming the entity, in this case the crowd. It takes into account the interactions of the individuals with their immediate surroundings, with other individuals, as well as their own motivation. The direction of motion, speed, and acceleration of the individual receive prime focus in the design. Collective behavior is assimilated from the knowledge of the motion of the individuals [5], [42]. On the contrary, the macroscopic model lays emphasis on group behavior, with the focus shifting from the individual to the entity as a whole [11], [33]. Rather than accounting for the individual interactions and the perturbations introduced by them, the macroscopic model gives a broader picture. There is emphasis now on average velocity, average acceleration and the general direction of motion [16], [26]. Individual behavioral effects and preferences are smoothed out giving a net average effect. Consequently, such modeling methods are prone to adapt an Eulerian approach of analysis involving the center of mass and density concepts [45].

Manuscript received August 20, 2014; accepted October 13, 2014. Date of publication January 26, 2015; date of current version May 13, 2015. This paper was recommended by Associate Editor G. Biswas.

The authors are with the Department of Electronics and Telecommunication Engineering, Jadavpur University, Kolkata 700032, India (e-mail: debdipta_goswami@yahoo.in).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSMC.2015.2389763

Interesting work in macroscopic analysis includes that of Zhang [53]. The model proposed by him was based on the microscopic-to-macroscopic derivation property. Incorporating partial differential equations that depict the continuity and momentum equations, the model could venture the anisotropic nature of traffic flows. This was in compliance with the empirical traffic flow behavior in which the pedestrian movement is influenced by the current and front positions and not so by rear positions. This behavior of crowds distinguishes itself from that of a fluid, where a fluid particle is influenced from all directions. Al-Nasur and Kachroo [38] extended the 1-D vehicle traffic flow model of Zhang [53] to a 2-D pedestrian movement flow. The model incorporated bidirectional flow and provided ways to control different pedestrian behaviors. The anisotropic feature in Zhang's model [53] was preserved even on extension to two dimensions. The system that occurs is a 2-D nonlinear hyperbolic partial differential equation system. Simulations were performed using numerical finite volume methods.

The model presented in this paper is a microscopic model and is based on the Lagrangian, rather than the Eulerian approach. Recent works have expressed increased focus on the individual and local interactions. Helbing [17] stressed the need of the agent-based models inasmuch as local coordinations come to light in such models. Also, there is no denying the fact that collective behavior results from each of those elementary individual interactions and hence their study will provide a better insight about collective crowd behavior. Helbing and Molnár's work [22] encircling around the social force model rightly espouses the flexibility in simulation of individual pedestrian motion. Inspired by the methods of Helbing [19], this paper depicts the acceleration of an individual in the crowd and takes into consideration its immediate surroundings, the interactions with the proximity groups, obstacles in its path and its intrinsic preferred motion. The social force concept gives a mathematical form to the systematic behavioral changes resulting from the interactions of individuals. Situations of panic in crowd dynamics find a significant place in Helbing *et al.*'s work [18], [21]. The force expressions incorporate additional terms in panic situations. There is a growing unrest among the individuals of the crowd, well justified by the extra frictional terms. Moussaïd *et al.* [36], [37] explored the group patterns in crowds where the level of interaction is midway between a macroscopic and a microscopic model. Group working patterns is a new direction given by their work. With increase in density of the group, the linear walking pattern changes to a V-like pattern. It presents an analysis where better communication among the individuals is taken care of.

There have been diverse works on crowding in the fields of computer vision [4], [13], [30], [39], [47], [50] and machine learning as well [1], [24]. Ali and Shah [3] proposed a framework based on Lagrangian particle dynamics that sought to segment high-density crowd flows and detect the aberrations in flow. Vision problems in this field may range from the crowd information extraction, recognition, or tracking [6], [27], [35]. Scovanner and Tappen [41] presented a method to learn the parameters governing pedestrian motion by observing video data. A continuous pedestrian cost model is optimized in their work and the results find use in real-life tracking situations.

The main distinction that this paper presents in this field of work is the Lagrangian approach to microscopic analysis of crowds with a precise mathematical formulation. The simplicity of the mathematical representation is resonant with the need for flexible simulation. This, however, does not in any way compromise with the inclusion of the essential interactive forces among the individuals. The rest of this paper is organized as follows. Section II gives a vivid analysis of the crowd dynamics model presented in this paper. Section III provides the stability analysis of the model. The stability analysis is based on the construction of the Lyapunov energy function [31], [43], [49] via the variable-gradient method. In Section IV, we have linearized the dynamics to mathematically express the small perturbations about the equilibrium points in terms of the parameters of the dynamics. The relevant simulations constitute Section V. Section VI discusses future work and concludes this paper.

II. Proposal of the Dynamics

The proposed system of crowd dynamics seeks to model the individuals forming the crowd by considering the forces that affect their motions. The crowd or pedestrian dynamics being essentially a 2-D modeling, take into account the position, velocity, and acceleration vectors, respectively, in $\mathcal{R}^2$. Let us consider N individuals forming the crowd. The acceleration of the ith individual in $\mathcal{R}^2$ is governed by the relation

$$\frac{dv_i}{dt} = (\omega - 1)v_i - \sum_{\substack{\forall j \\ j\neq i}} C_{aij}\left(a\frac{x_j - x_i}{\|x_i - x_j\|^{m+1}} - b\frac{x_j - x_i}{\|x_i - x_j\|^{n+1}}\right) - \left[\sum_{\substack{\forall j \\ j\neq i}} C_{rij}\exp\left(-\|x_i - x_j\|^2\right)\frac{x_j - x_i}{\|x_i - x_j\|} + \frac{d_{\text{wall}} - x_i}{\|x_i - d_{\text{wall}}\|^3}\right] \tag{1}$$

where ω is positive-definite, C_{aij} and C_{rij} are positive semi-definite real constants, C_{aij} and C_{rij} being defined as follows:

$$C_{aij} = \begin{cases} 0 & \text{if } j \text{ does not form a group with } i \\ +ve & \text{if } j \text{ forms a group with } i \end{cases}$$

and

$$C_{rij} = \begin{cases} +ve & \text{if } j \text{ does not form a group with } i \\ 0 & \text{if } j \text{ forms a group with } i \end{cases}$$

x_i's and v_i's are all 2-D vectors and the vector notation is implicitly assumed. m and n are positive powers determining the intragroup attractant-repellent profile.

The acceleration being directly related to the forces that influence the motion of the individuals in the crowd, we can now highlight the various forces that dictate the motion. There is an interaction among the individuals of the crowd in the course of their motion, thus inclining the individuals to form groups. The pedestrian group formation is one aspect that gives the Lagrangian crowd model a cooperative form. This unique feature among crowds distinguishes it clearly from swarms or other multiagent systems whose motion is governed by sharing of information and cooperation among all the agents [2], [48]. However, among pedestrians, more often than not, there is group formation whereby acquaintances move together. There exists a visible distinction between a group of friends or family members and between two such groups which are not acquainted with each other. This social behavior observed among crowds is exploited in the dynamics. The intragroup attractant-repellent profile is modeled in the likes of the forces that operate in crystal structures. The Lennard-Jones potential is one such model that defines the interaction between a pair of neutral atoms or molecules. The Pauli repulsion at short distances and the attractive van der Waals forces at long range come into play. In a group of pedestrians, this interplay of attraction and repulsion finds prominence owing to the fact that each individual has a radius of comfort. If the force was only attractive, then within a group, the distance between individuals would reduce to zero and a social behavior would seem anti-social, even hostile. The profile of the attraction and repulsion within a group member can simply be given by the equation

$$f(x) = C_a\left(a\frac{1}{|x|^m} - b\frac{1}{|x|^n}\right) \quad (2)$$

$f(x)$ provides a strong repulsion at close enough distances. We define $r = r_0$ as the radius of comfort, where the attractive and repulsive forces are equal. At distances less that the radius of comfort, strong repulsion starts acting that prevents the hostile situation within a group. This further allows the representation of the individuals as a finite radius disc, since r tending to zero implies $f(x)$ tending to infinity. Each individual can be represented as a disc of radius r_i, where $r_i \in (0, r_0)$ and $r_i \ll r_0$.

The attraction starts at $r > r_0$ and has the maximum value at $r = r_m$. As the distance increases indefinitely, there no longer remains any attraction as even acquaintances then fail to recognize each other's presence owing to the great distance of separation. This is in compliance with everyday experience. Even if two individuals are known to each other, they may not form a group in a busy street or in that case, say, a market place, if they are separated by a distance that hinders the establishment of any interaction between them. The interaction is possible when they are at an eyesight distance from each other.

The constants C_{aij} further scale the attractant-repellent profile. It may have the same value irrespective of the presence of individuals. It may even assume different values based on how strong or how weak the interaction is between two members in the crowd. We have kept flexibility in the parameter choice of the dynamics based on the domain of application. The constants a, b, m, and n determine the nature of $f(x)$. We have shown the influence of these constants on the profile by providing plots of $f(x)$ with their variation.

The plot of this relation for different values of C_a is shown in Fig. 1. C_a scales $f(x)$ and the functional value at minima changes with its variation. But the radius of comfort r_0 as well as the distance at which maximum attraction is obtained, r_m, remain unchanged. The variation of $f(x)$ with b is shown

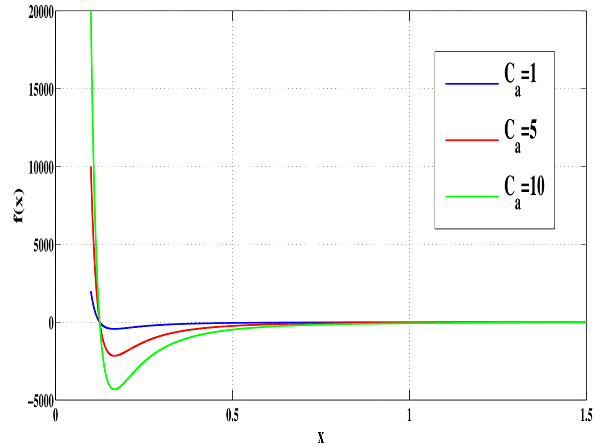

Fig. 1. $f(x)$ versus x for the variation of C_a with $a = 1$ and $b = 8$.

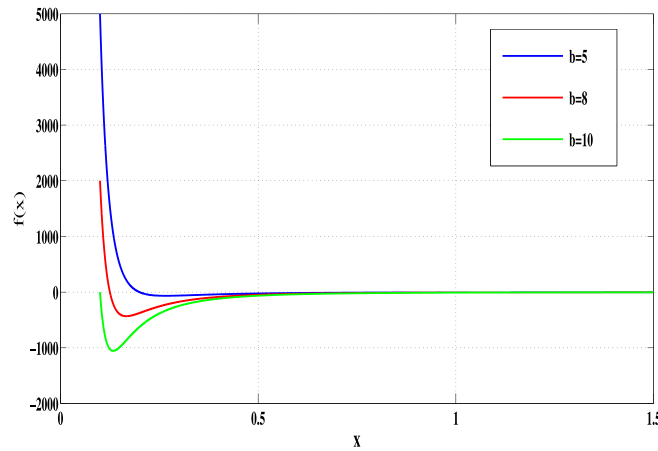

Fig. 2. $f(x)$ versus x for the variation of b with $a = 1.0$ and $C_a = 1$.

in Fig. 2. Here, with increase in b, the radius of comfort decreases, but the maximum attractive force increases. For both the plots, we took $m = 12$ and $n = 6$.

The repulsion between disparate groups and strangers is taken into account by the repulsive exponential terms in the dynamics. The repulsion varies as the function $\exp(-\|x_i - x_j\|^2)$. As the distance between the individuals increases, the repulsion decreases. This relation is again in direct analogy with the social scenario where the repulsive forces are negligible when there is a considerable distance between two unacquainted groups. However, if one group comes in close affinity of another, there is the presence of strong repulsion so that the merging of distinct groups does not take place and each group retains its own cooperative identity. It is worthy now to mention the modeling of repulsion with the exponential variation, rather than in any other form. The motivation for such exponential variation arises from the fact that exponential repulsive forces fall off faster than their attractive counterparts. When the repulsion between like charges and attraction between unlike charges occurs in a crystal, it gives rise to an attractant-repellent profile that attains equilibrium state with the rapid decrease of the repulsive force with distance compared with the attractive force. Consequently, our proposed model considers an exponential variation for the repulsion. For repulsion, likewise, the value of parameter C_{rij} depends on the degree of repulsion. It may be identical for all interactions or it may be varied based on the requirements of application.

Apart from the interactions with other individuals in the crowd, we consider the ability of the pedestrian to avoid obstacles in his path. The obstacle may be a wall, a dead end, or any other object that needs to be avoided in the course of motion. The term $(d_{\text{wall}} - x_i)/\|x_i - d_{\text{wall}}\|^3$ takes into account the effect of obstacles. The obstacle has negligible influence on the motion when it is far away from the pedestrian. However, there is a strong retarding force inhibiting direct collision with the obstacle when the pedestrian is in its vicinity. Discretizing the relation for acceleration with respect to time gives $v_i(t+1)$ as a function of $v_i(t)$ along with the other terms. The intrinsic ability of the individual to move on his own discretion is reflected by the term ωv_i. This inertia factor ω defines the self motivation which is pivotal in the modeling of a crowd dynamics, a departure from the modeling of inanimate entities. It may so happen that one member of a group may find the need to differ his direction of motion from that of the group. It represents the will of the member to manoeuvre his motion as per his necessities. The confluence of these various forces and motivations for movement observed in crowds forms the dynamics in its entirety. It is fascinating to note how with minimal changes to the basic dynamics, we are able to incorporate other salient features characterizing the motion of individuals in a crowd.

The crowd may be seen as groups of individuals moving toward a destination. To model the pedestrian and crowd system within a certain time interval in a 2-D field, we make certain assumptions valid at a large scale. Firstly, each individual in a crowd system originates from a source and reaches a sink or destination. This is similar to a pedestrian starting his/her journey from the source and reaching the desired destination. After reaching the destination, the corresponding individual's velocity becomes zero abruptly. We enforce C_{ai} and C_{ri} to be zero when the individual is at rest so that it does not influence the motion of any other individual. If the individual again starts his motion, C_{ai} and C_{ri} regain nonzero values, so does its velocity, thereby incorporating it into the system. The second assumption is that within a certain time interval, for large scale statistical data, we can say that the number of individuals reaching the destination is the same as the number originating from the source. The dynamics is able to model the motion of the pedestrian or crowd system with individuals moving from source to destination. The source may be unknown or may not come under the purview of modeling concerns. The destination may be a shop in a market place, the work place, a park, or even a restaurant for a pedestrian. Once the group, or even a single individual reaches such a destination, his motion ceases. The velocity decreases to zero on reaching the desired place. This real-life situation is aptly portrayed by our dynamics where the velocities of the groups converge to zero, indicating that the destination has been reached. For the purpose of simulations, at the starting time, the position, and velocity of the individuals may be chosen to be random and the interplay of the forces becomes clear when the motion starts. There is no necessity in a crowd dynamics, however, for position convergence to a particular point, for even though the group members reach the same velocity, they are not at the same coordinates at the same instant of time. Also owing to the intrinsic motivation, one member may be walking ahead in the group, others may be trailing. As long as they are at an eyesight distance and the attraction remains, they retain their group identity.

The velocity converging to zero can be considered a specific scenario. Although this situation does arise, the groups retaining their constant velocities is equally observed. Let us take the example of a group of people taking the exit route after the train reaches the station. Here the velocities does not reach zero, rather all individuals in a particular group after exit maintain a velocity, which may be different from individuals in other groups. Other instances of this phenomenon occur in places where there is an urge for continual flow after an intermediate transit. The intermediate transit may be the ticket check or the exit door. We have been able to include this behavior in our dynamics. The acceleration in such a case involves an additional term. If there are M groups and the ith individual belongs to group m, where the desired group velocity is v_{gm}, its acceleration is given by

$$\frac{dv_i}{dt} = (\omega - 1)v_i - \sum_{\substack{\forall j\\ j\neq i}} C_{aij}\left(a\frac{x_j - x_i}{\|x_i - x_j\|^{m+1}} - b\frac{x_j - x_i}{\|x_i - x_j\|^{n+1}}\right) - \left[\sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|x_i - x_j\|^2\right)\frac{x_j - x_i}{\|x_i - x_j\|} + \frac{d_{\text{wall}} - x_i}{\|x_i - d_{\text{wall}}\|^3}\right] + K(v_{gm} - v_i) \tag{3}$$

where $K \in (0, 1)$. This modification makes it possible to extend our dynamics even for a flocking system, where the velocity of the entire swarm approaches a definite value. Such flocking systems similarly attain the steady velocity with time through cooperation between all the entities. Nevertheless, there is a slight difference with the crowd as regards the different velocities that different groups can attain.

A. Panic in Crowds and Its Modeling

The multitude of crowd disasters that frequent public places deserves special concern. Such tragic incidents may be the result of casualties, man-made or natural, or they may be triggered by sheer negligence such as the rush of people to the exit of a public place, shrine, stadium, railway station, subway, airports, and the like [51]. We seek to address such situations through modeling of the panic in crowds. Such a modeling would enable the design of intelligent evacuation procedures, better conceptualization of the emergency situations and an insight into precautionary measures to avoid future life-threatening incidents. Rather than providing an empirical or statistical analysis, we mathematically model the phenomenon by introducing a term in the basic dynamics that will aid to simulate the urgent egress in a panic situation. Before, we proceed to the dynamics, it is necessary to clarify panic in crowds. By panic, we refer to situations where there is an increasing density of people and the cooperative group structure illustrated before gives way to a repulsive, impetuous, instinctive action. It is merely individualistic decisions that govern the motion of people in such situations. For instances of panic in crowds, we can picture public places, say a theater hall, where at the conclusion of a play, people rush out of the hall. The large group of people moving out of a railway station, the frenetic movement of fervent supporters at a stadium, or the extreme situation of people escaping from a building on fire—all present varying situations of panic. This leads us to introduce the degree of panic that helps demarcate one situation from another. Accurate modeling necessitates such a distinction as rescue measures for a panic situation of lower degree may not be identical to one with a greater degree of panic and requiring urgent evacuation measures. The degree of panic, denoted in our dynamics by n, indicates the extremity of the turbulence in the high-density crowd issuing forth. The previous examples of the theater hall and the building on fire elucidate it further. In case of the theater hall, there is no doubt an urge for the exit among the people. Nonetheless, some people still take time, discuss the plot of the play and slowly move toward the exit. The crowding at the egress is to a lesser extent. On the contrary, the fire presents an extremity where there is a life and death possibility. Evidently, the turbulence and the crowding are to a greater extent. Masoud [32] discussed the only two ways in which conflict can arise—the radial and tangential forces. We have discussed the radial attraction and repulsion in the dynamics. In situations of panic, when the individuals are in close affinity and have bodily contacts, physical interaction forces, like sliding friction forces come to play. If $d_{i,j} < r_{i,j} = r_i + r_j$, where $d_{i,j}$ is the distance between x_i and x_j and r_i is the radius of the ith pedestrian, the tangential friction force $f_{i,j}^{fr} = \kappa \Theta(r_{i,j} - d_{i,j}) \Delta v_{j,i}^t \hat{t}_{i,j}$ comes into picture, where $\Theta(z) = z$ if $z > 0$ and 0 otherwise. This impends the relative tangential motion of the pedestrians and is proportional to their tangential relative velocity $\Delta v_{j,i}^t$. Accordingly, the dynamics representing panic in crowds is given by

$$\begin{aligned}\frac{dv_i}{dt} &= (\omega - 1)v_i - \sum_{\substack{\forall j\\ j\neq i}} C_{aij}\left(a\frac{x_j - x_i}{\|x_i - x_j\|^{m+1}} - b\frac{x_j - x_i}{\|x_i - x_j\|^{n+1}}\right)\\ &\quad - \left[\sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|x_i - x_j\|^2\right)\frac{x_j - x_i}{\|x_i - x_j\|}\right.\\ &\quad \left. + \frac{d_{\text{wall}} - x_i}{\|x_i - d_{\text{wall}}\|^3}\right] + \frac{C_{\text{evac}}}{\|x_i - x_{\text{evac}}\|^{(n+1)}}\cdot \hat{r}_{\text{evac}}\\ &\quad + \sum_{\substack{\forall j\\ j\neq i}} f_{i,j}^{fr}\end{aligned} \tag{4}$$

where the unit vector $\hat{r}_{\text{evac}}$ is given by

$$\hat{r}_{\text{evac}} = \frac{x_{\text{evac}} - x_i}{\|x_i - x_{\text{evac}}\|}$$

$n \in (0, 1)$ and C_{evac} is a positive constant. As n approaches 0, the degree of panic increases. Consequently, there is a greater rush toward x_{evac}. It is interesting to note that if n equals 1, the additional term is similar to the attraction terms that exist with the other individuals of the group. Thus, the introduction of the term signifying panic may be seen as simulating the sudden rush toward the evacuation coordinates that disintegrates the cooperative attractant-repellent profile and emulates the instinctive urge for safety and survival.

The parameter choice for a specific practical domain may be perfected by following a procedure similar to that suggested by Helbing and Johansson [19]. The error reduction by comparing between video tracking images and the motion simulated by the dynamics would lead to the best choice. The stability analysis provides further insight for a general parameter choice. Under any given circumstance, once the best fit is determined, subsequent decisions either on evacuation or traffic management can be taken from the simulations following the dynamics. Emergency situations may be better studied. Possible aberrations may be taken care of, and decision making can be improved via the predictions from the simulations. In the absence of such mathematical modeling, it would be immensely difficult to predict and design safety routes for disaster situations.

Modeling of panic and the rush towards the evacuation point further helps visualize the severity of the imminent crowd disasters. It would aid in deciding the dimensions of evacuation structures, explore alternate sites for evacuation and initiate intelligent crowd control in confined places. Since the simulations can determine the number of people crossing the urgent egress at different instants of time, it is possible to determine whether there will be congestion at the exit. This in turn will help design the egress for a certain maximum number of people depending on the crowd density at the particular public place so that congestion can be avoided even under the worst of circumstances. Successful implementation of the dynamics for efficient decision making in high-density crowding sites will lead to saving millions of lives in future.

III. Analyzing the Stability of the System

As we have seen earlier, our proposed dynamics is given by

$$\begin{aligned}\frac{dv_i}{dt} &= (\omega - 1)v_i - \sum_{\substack{\forall j\\ j\neq i}} C_{aij}\left(a\frac{x_j - x_i}{\|x_i - x_j\|^{m+1}} - b\frac{x_j - x_i}{\|x_i - x_j\|^{n+1}}\right)\\ &\quad - \left[\sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|x_i - x_j\|^2\right)\frac{x_j - x_i}{\|x_i - x_j\|} + \frac{d_{\text{wall}} - x_i}{\|x_i - d_{\text{wall}}\|^3}\right]\\ &= f(x_i, v_i)\end{aligned} \tag{5}$$

where vector notation is implicitly assumed.

To check for the stability of the system we intuitively construct the Lyapunov energy function (6), shown at the top of the next page.

For $L(x_i, x_j, v_i, v_j)$ to be a Lyapunov energy function, we first take note of the fact that:

1) *Value at Critical Point(s):* $L(0, 0, 0, 0) = 0$.

$$L\left(x_i, x_j, v_i, v_j\right) = -(\omega-1)\sum_{\forall i}\int_0^{v_i}\eta_i\, d\eta_i + (\omega-1)\sum_{\forall i}\int_0^{x_i}\left[\sum_{\substack{\forall j\\ j\neq i}} -C_{aij}\left(a\frac{x_j-x_i}{\|x_i-x_j\|^{m+1}} - b\frac{x_j-x_i}{\|x_i-x_j\|^{n+1}}\right) - \left(\sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|\eta_i-x_j\|^2\right)\frac{x_j-\eta_i}{\|\eta_i-x_j\|} + \frac{d_{\text{wall}}-\eta_i}{\|\eta_i-d_{\text{wall}}\|^3}\right)\right]d\eta_i \quad (6)$$

$$v_i + \sum_{\substack{\forall j\\ j\neq i}} C_{aij}\left(b\frac{x_j-x_i}{\|x_i-x_j\|^{n+1}}\right) > \sum_{\substack{\forall j\\ j\neq i}} C_{aij}\left(a\frac{x_j-x_i}{\|x_i-x_j\|^{m+1}}\right) + \left[\sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|x_i-x_j\|^2\right)\frac{x_j-x_i}{\|x_i-x_j\|} + \frac{d_{\text{wall}}-x_i}{\|x_i-d_{\text{wall}}\|^3}\right] \quad (7)$$

2) *Partial Derivatives:* The partial derivatives $\frac{\partial L}{\partial x_i}$, $\frac{\partial L}{\partial x_j}$, $\frac{\partial L}{\partial v_i}$, $\frac{\partial L}{\partial v_j}$ exist.
3) *Value at Other Points:* Further $L(x_i, x_j, v_i, v_j)$ will be greater than zero for $x_i, x_j, v_i, v_j \neq 0$ if

$$v_i - \sum_{\substack{\forall j\\ j\neq i}} C_{aij}\left(a\frac{x_j-x_i}{\|x_i-x_j\|^{m+1}} - b\frac{x_j-x_i}{\|x_i-x_j\|^{n+1}}\right) - \left[\sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|x_i-x_j\|^2\right)\frac{x_j-x_i}{\|x_i-x_j\|} + \frac{d_{\text{wall}}-x_i}{\|x_i-d_{\text{wall}}\|^3}\right] > 0$$

i.e., function (7), shown at the top of the page.

This relates to the physical situation where if the intrinsic velocity and the attraction between the individuals of the particular group is greater than the repulsion with other groups, then the Lyapunov energy function is positive definite and feasible for further stability analysis. This condition indeed sheds some light on parameter choice of the dynamics. As already illustrated by the variation of $f(x)$ with m, n, a, and b, for the attraction to be more, the difference between m and n should be large with $m > n$. b having a larger value than a increases the attraction. Also C_{aij} being of larger magnitude than C_{rij} ensures that the intragroup attraction dominates the intergroup repulsion. The condition for stability and the physical scenario are in complete consonance as the attraction is vital for retaining group identity.

Now, to test the stability of the system, we differentiate (6) under the integral sign to obtain

$$\frac{dL}{dt} = \frac{\partial L}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial L}{\partial v_i}\frac{dv_i}{dt} = -(\omega-1)\sum_{\forall i} v_i\frac{dv_i}{dt} + (\omega-1)\sum_{\forall i}\left[\sum_{\substack{\forall j\\ j\neq i}} -C_{aij}\left(a\frac{x_j-x_i}{\|x_i-x_j\|^{m+1}} - b\frac{x_j-x_i}{\|x_i-x_j\|^{n+1}}\right) - \sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|x_i-x_j\|^2\right)\frac{x_j-x_i}{\|x_i-x_j\|} + \frac{d_{\text{wall}}-x_i}{\|x_i-d_{\text{wall}}\|^3}\right]\frac{dx_i}{dt}. \quad (8)$$

We recognize that dx_i/dt is nothing but v_i to write

$$\frac{dL}{dt} = -(\omega-1)\sum_{\forall i} v_i\frac{dv_i}{dt} + (\omega-1)\sum_{\forall i} \times\left[\sum_{\substack{\forall j\\ j\neq i}} -C_{aij}\left(a\frac{x_j-x_i}{\|x_i-x_j\|^{m+1}} - b\frac{x_j-x_i}{\|x_i-x_j\|^{n+1}}\right) - \sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|x_i-x_j\|^2\right)\frac{x_j-x_i}{\|x_i-x_j\|} + \frac{d_{\text{wall}}-x_i}{\|x_i-d_{\text{wall}}\|^3}\right]v_i. \quad (9)$$

Equivalently we have (10), shown at the top of the next page. From (5), we recognize the term in the square brackets to be nothing but $(\omega-1)\,v_i$. Hence

$$\frac{dL}{dt} = -(\omega-1)\sum_{\forall i} v_i((\omega-1)\,v_i) = -(\omega-1)^2\sum_{\forall i} v_i^2 < 0.$$

This thus implies that the given dynamics is asymptotically stable under condition (7).

IV. Linearization of the Dynamics

At this juncture, we would like to linearize the dynamics of the crowd system that we have stated and carry out a detailed analysis. The linearization facilitates the study of sensitivity where the behavior of the system can be clarified under the influence of a small perturbation around its fixed points. For the sake of clarity and mathematical lucidity we assume an entirely 1-D inspection into the system. The system dynamics for the ith particle is given by

$$\frac{dv_i}{dt} = (\omega-1)v_i + \sum_{\substack{\forall j\\ j\neq i}} C_{aij}\left(a\frac{x_j-x_i}{\|x_i-x_j\|^{m+1}} - b\frac{x_j-x_i}{\|x_i-x_j\|^{n+1}}\right) - \left[\sum_{\substack{\forall j\\ j\neq i}} C_{rij}\exp\left(-\|x_i-x_j\|^2\right)\frac{x_j-x_i}{\|x_i-x_j\|} + \frac{d_{\text{wall}}-x_i}{\|x_i-d_{\text{wall}}\|^3}\right] = f(x_i, v_i). \quad (11)$$

$$\frac{dL}{dt} = -(\omega - 1)\sum_{\forall i} v_i \left[\frac{dv_i}{dt} - \left(\sum_{\substack{\forall j \\ j\neq i}} -C_{aij}\left(a\frac{x_j - x_i}{\|x_i - x_j\|^{m+1}} - b\frac{x_j - x_i}{\|x_i - x_j\|^{n+1}}\right) - \sum_{\substack{\forall j \\ j\neq i}} C_{rij}\exp\left(-\|x_i - x_j\|^2\right)\frac{x_j - x_i}{\|x_i - x_j\|} + \frac{d_{\text{wall}} - x_i}{\|x_i - d_{\text{wall}}\|^3}\right)\right] \tag{10}$$

For 1-D d, this reduces to

$$\frac{dv_i^d}{dt} = (\omega - 1)v_i^d + \sum_{\substack{\forall j \\ j\neq i}} C_{aij}\left(a\frac{x_j^d - x_i^d}{\|x_i^d - x_j^d\|^{m+1}} - b\frac{x_j^d - x_i^d}{\|x_i^d - x_j^d\|^{n+1}}\right) - \left[\sum_{\substack{\forall j \\ j\neq i}} C_{rij}\exp\left(-\|x_i^d - x_j^d\|^2\right)\frac{x_j^d - x_i^d}{\|x_i^d - x_j^d\|} + \frac{d_{\text{wall}} - x_i^d}{\|x_i^d - d_{\text{wall}}\|^3}\right] = f\left(x_i^d, v_i^d\right). \tag{12}$$

For 1-D, $\|x_i - x_j\|$ is given simply as $|x_i - x_j|$. Further clarity is introduced if we drop the superscripts to get

$$\frac{dv_i}{dt} = (\omega - 1)v_i + \sum_{\substack{\forall j \\ j\neq i}} C_{aij}\left(a\frac{x_j - x_i}{|x_i - x_j|^{m+1}} - b\frac{x_j - x_i}{|x_i - x_j|^{n+1}}\right) - \left[\sum_{\substack{\forall j \\ j\neq i}} C_{rij}\exp\left(-|x_i - x_j|^2\right)\frac{x_j - x_i}{|x_i - x_j|} + \frac{d_{\text{wall}} - x_i}{|x_i - d_{\text{wall}}|^3}\right] = f(x_i, v_i). \tag{13}$$

This can be simplified to

$$\frac{dv_i}{dt} = (\omega - 1)v_i - \sum_{\substack{\forall j \\ j\neq i}} C_{aij}\left(a\frac{1}{\left(x_i - x_j\right)^{m-1}|x_i - x_j|} - b\frac{1}{\left(x_i - x_j\right)^{n-1}|x_i - x_j|}\right) - \left[\sum_{\substack{\forall j \\ j\neq i}} C_{rij}\exp\left(-\left(x_i - x_j\right)^2\right)\cdot\frac{x_j - x_i}{|x_i - x_j|} - \frac{1}{(x_i - d_{\text{wall}})|x_i - d_{\text{wall}}|}\right] = f(x_i, v_i). \tag{14}$$

Suppose that the operating point of the equation is at $x_i = x_i^*$, $v_i = v_i^*$ and there are small perturbations of δx_i and δv_i about the operating point. Since the dynamics is a function of x_i and v_i only, we may as well write

$$f\left(x_i^* + \delta x_i, v_i^* + \delta v_i\right) = \frac{d}{dt}\left(v_i^* + \delta v_i\right) = \frac{d(\delta v_i)}{dt}$$

since all derivatives vanish at the operating point. Attempting a Taylor series expansion of the function $f(x_i^* + \delta x_i, v_i^* + \delta v_i)$ and retaining only the first-order derivatives gives

$$f\left(x_i^* + \delta x_i, v_i^* + \delta v_i\right) = f\left(x_i^*, v_i^*\right) + \left.\frac{\partial f}{\partial x_i}\right|_{\substack{x = x_i^* \\ v = v_i^*}}\delta x_i + \left.\frac{\partial f}{\partial v_i}\right|_{\substack{x = x_i^* \\ v = v_i^*}}\delta v_i. \tag{15}$$

This implies that

$$\frac{d(\delta v_i)}{dt} = \left((\omega - 1)v_i^* - \sum_{\substack{\forall j \\ j\neq i}} C_{aij}\left(a\frac{1}{\left(x_i^* - x_j\right)^{m-1}|x_i^* - x_j|} - b\frac{1}{\left(x_i^* - x_j\right)^{n-1}|x_i^* - x_j|}\right) - \left[\sum_{\substack{\forall j \\ j\neq i}} C_{rij}\exp\left(-\left(x_i^* - x_j\right)^2\right)\cdot\frac{x_j - x_i^*}{|x_i^* - x_j|} - \frac{1}{\left(x_i^* - d_{\text{wall}}\right)|x_i^* - d_{\text{wall}}|}\right]\right) + \left(\sum_{\substack{\forall j \\ j\neq i}} C_{aij}\left[\frac{m}{\left(x_i^* - x_j\right)^m|x_i^* - x_j|} - \frac{n}{\left(x_i^* - x_j\right)^n|x_i^* - x_j|}\right] - \left[\sum_{\substack{\forall j \\ j\neq i}} -2C_{rij}\left(x_i^* - x_j\right)\exp\left(-\left(x_i^* - x_j\right)^2\right)\cdot\frac{x_j - x_i^*}{|x_i^* - x_j|} + \frac{2}{\left(x_i^* - d_{\text{wall}}\right)^2|x_i^* - d_{\text{wall}}|}\right]\right)\delta x_i + (\omega - 1)\delta v_i. \tag{16}$$

Taking Laplace transforms of both sides of the above equation gives

$$s\delta v_i(s) = \frac{\Gamma}{s} + (\omega - 1)\delta v_i(s) - \Delta\delta x_i(s) \tag{17}$$

where the constants Γ and Δ are given by

$$\Gamma = (\omega - 1)v_i^* - \sum_{\substack{\forall j \\ j \neq i}} C_{aij} \left(a \frac{1}{(x_i^* - x_j)^{m-1} |x_i^* - x_j|} - b \frac{1}{(x_i^* - x_j)^{n-1} |x_i^* - x_j|} \right) - \left[\sum_{\substack{\forall j \\ j \neq i}} C_{rij} \exp\left(-\left(x_i^* - x_j\right)^2 \right) \cdot \frac{x_j - x_i^*}{|x_i^* - x_j|} - \frac{1}{\left(x_i^* - d_{\text{wall}}\right) |x_i^* - d_{\text{wall}}|} \right] \tag{18}$$

and

$$\Delta = -\left(\sum_{\substack{\forall j \\ j \neq i}} C_{aij} \left[\frac{m}{(x_i^* - x_j)^m |x_i^* - x_j|} - \frac{n}{(x_i^* - x_j)^n |x_i^* - x_j|} \right] - \left[\sum_{\substack{\forall j \\ j \neq i}} -2C_{rij} \left(x_i^* - x_j\right) \exp\left(-\left(x_i^* - x_j\right)^2 \right) \cdot \frac{x_j - x_i^*}{|x_i^* - x_j|} + \frac{2}{\left(x_i^* - d_{\text{wall}}\right)^2 |x_i^* - d_{\text{wall}}|} \right] \right). \tag{19}$$

We rearrange this to get

$$(s - (\omega - 1))\delta v_i(s) = \frac{\Gamma}{s} - \Delta \delta x_i(s)$$

or, equivalently

$$s(s - (\omega - 1))\delta v_i(s) + s\Delta\delta x_i(s) = \Gamma. \tag{20}$$

Again, we know that

$$v_i = \frac{dx_i}{dt}.$$

Hence

$$v_i^* + \delta v_i = \frac{d}{dt}(x_i^* + \delta x_i) = \frac{d(\delta x_i)}{dt}.$$

Again, Laplace Transforms of both sides of this equation yields

$$s\delta x_i(s) = \frac{v_i^*}{s} + \delta v_i(s)$$

which is the same as

$$-s\delta v_i(s) + s^2 \delta x_i(s) = v_i^*. \tag{21}$$

Solving (20) and (21), we get the expressions for $\delta x_i(s)$ and $\delta v_i(s)$ to be

$$\delta x_i(s) = \frac{\Gamma + v_i^*(s - (\omega - 1))}{s\{s(s - (\omega - 1)) + \Delta\}} \tag{22a}$$

$$\delta v_i(s) = \frac{s\Gamma - v_i^* \Delta}{s\{s(s - (\omega - 1)) + \Delta\}}. \tag{22b}$$

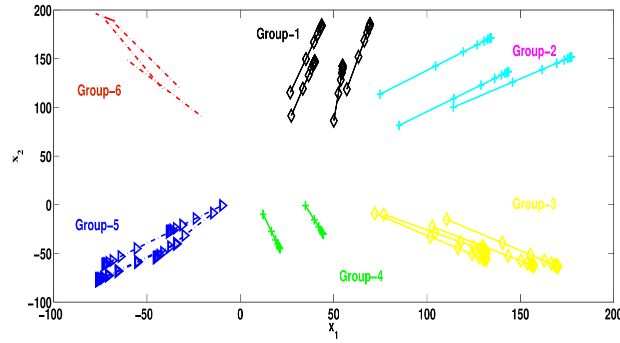

Fig. 3. Position phase-plot for six groups moving simultaneously in a 2-D field.

This expresses the incremental changes in $\delta x_i(s)$ and $\delta v_i(s)$ as a function of the inertial factor ω. The linearization seeks to describe the behavior of the dynamical system near the equilibrium points. Furthermore, the nature of the equilibrium points can be determined from the eigen values of the Jacobian matrix at the equilibrium point. However, it is important to note here that the linearization does not emulate the complex behavior of the nonlinear dynamics. It just gives the small perturbations about the equilibrium a mathematical form and relates them to the parameters of the dynamics.

V. Machine Simulation

The simulations of the proposed crowd model attempt to illustrate the crowd behavior discussed earlier. The group identity in pedestrians, the velocity convergence in a group, avoidance of obstacles, and rush toward the evacuation point in panic situations are all taken up in this section.

Fig. 3 shows a 2-D plot of the positions when 20 individuals form six groups. The simulations have been performed with the parameters $a = 1$, $b = 8$, $m = 12$, $n = 6$, $C_a = 1.5$, and $C_r = 1.0$. The dynamics is stable and is able to represent the crowd behaviour in accordance with the real-world situation for even large variations in the parameters. The parameter choice has been based on the requirement of stability of the Lyapunov function considering the fact that the intragroup attraction should be greater than the intergroup repulsion. The value of b has been kept larger than a, C_a is greater in magnitude than C_r and m, n take the values as in Lennard–Jones potential or $12 - 6$ potential. Here, we see that the velocities of the individuals in each group converge to a common value and after convergence, different groups may move with different velocities. This supports the inference made in Section II about group formation in crowds and their subsequent motion.

Fig. 4 shows the 2-D velocity phase-plot where the velocity converges to zero when the individuals reach their respective destinations. However, it is not necessary that all the groups reach their destination simultaneously. Again, the groups may move after their members reach a common velocity and in that case, the velocities of the members of different groups will converge to different values unique for a group. This velocity, as mentioned in Section II as desired group-velocity,

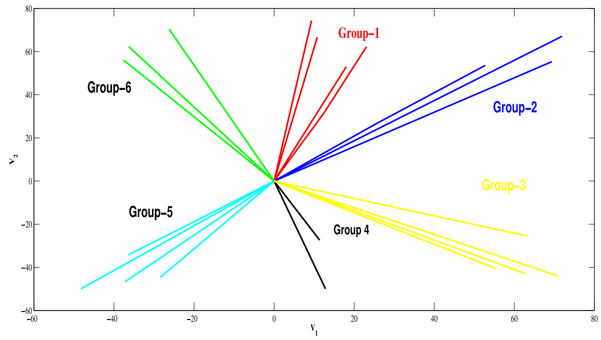

Fig. 4. Velocity phase-plot for six groups moving simultaneously in a 2-D field.

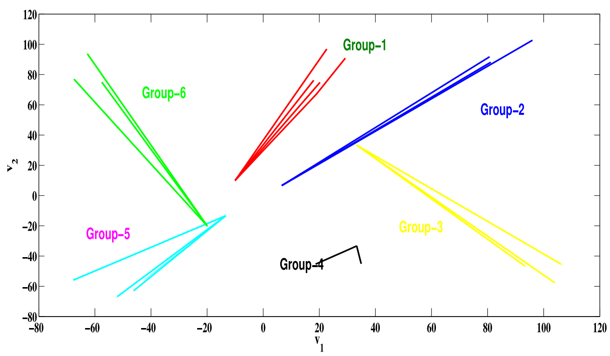

Fig. 5. Velocity phase-plot for six groups moving simultaneously in a 2-D field. Velocities of the six groups converge to different values.

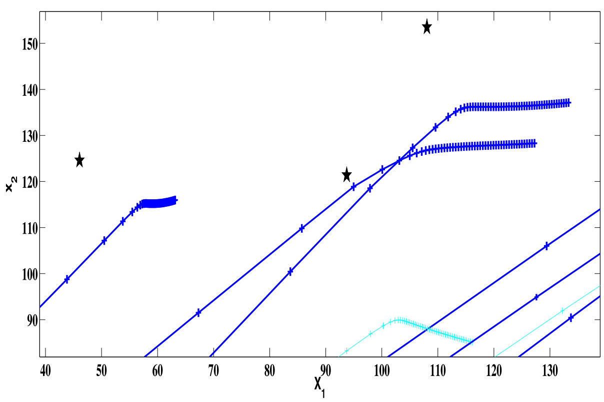

Fig. 6. Trajectories of pedestrians bend near obstacles.

will create a flocking-type behavior in the system and is shown in Fig. 5.

The pedestrians change their course of motion when there is an obstacle present. Fig. 6 shows the bending of the trajectories to avoid the obstacles. The obstacles are marked with a star.

Figs. 7 and 8 deal with the situation when panic arises. The degree of panic is kept at $n = 0.1$ and 0.2 and the motion of five individuals in the system is plotted. The evacuation point (i.e., the exit or the elevator) is marked with a star. The figure shows that the individuals steadily move toward the evacuation

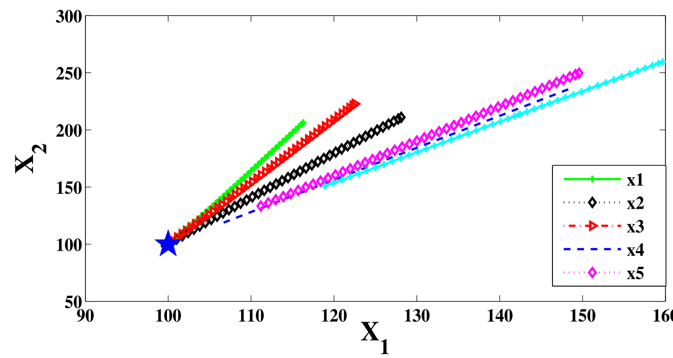

Fig. 7. Successive position plot for evacuation situation with degree of panic 0.1 and $C_{\text{evac}} = 1.5$. The evacuation point is marked by a star.

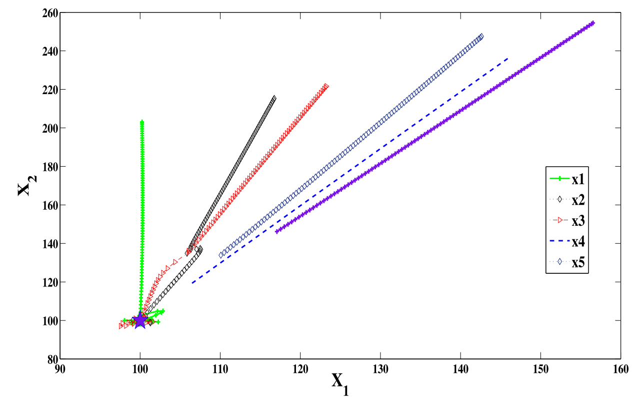

Fig. 8. Successive position plot for evacuation situation with degree of panic 0.2 and $C_{\text{evac}} = 1.5$. The evacuation point is marked by a star.

point. They can move even after reaching the point as people usually rush forward after the exit and do not stagnate there.

The evacuation procedure is further illustrated in Fig. 9 where a crowd consisting of 15 individuals, trying to escape through a single exit is shown. The positions of the individuals during different instances are plotted.

Finally, we have included some instances from real-life illustrating the group identity in pedestrians, situations of panic and rush toward the evacuation point. Snapshots from a video in Fig. 10 show how the pedestrians move in groups and retain their group identities. In the entire course of motion in the video snaps, the two groups, one consisting of four members and the other with two members function as distinctive units. There is velocity convergence; a particular radius of comfort exists; stable forces continue to operate and separation from other groups remain. All the features we had discussed in Section II about pedestrian behavior come to prominence in these snapshots. The simulations from our dynamics is thus in resonance with the real-world crowd and pedestrian behavior. Fig. 9 brings to light the aspect of evacuation and panic situations in crowds. In such high-density crowding situations, the frictional forces come into play. The rush toward the evacuation point is evident from the simulations. All these complete the link between crowding in the real world, its modeling by our dynamics and the relevant simulations.

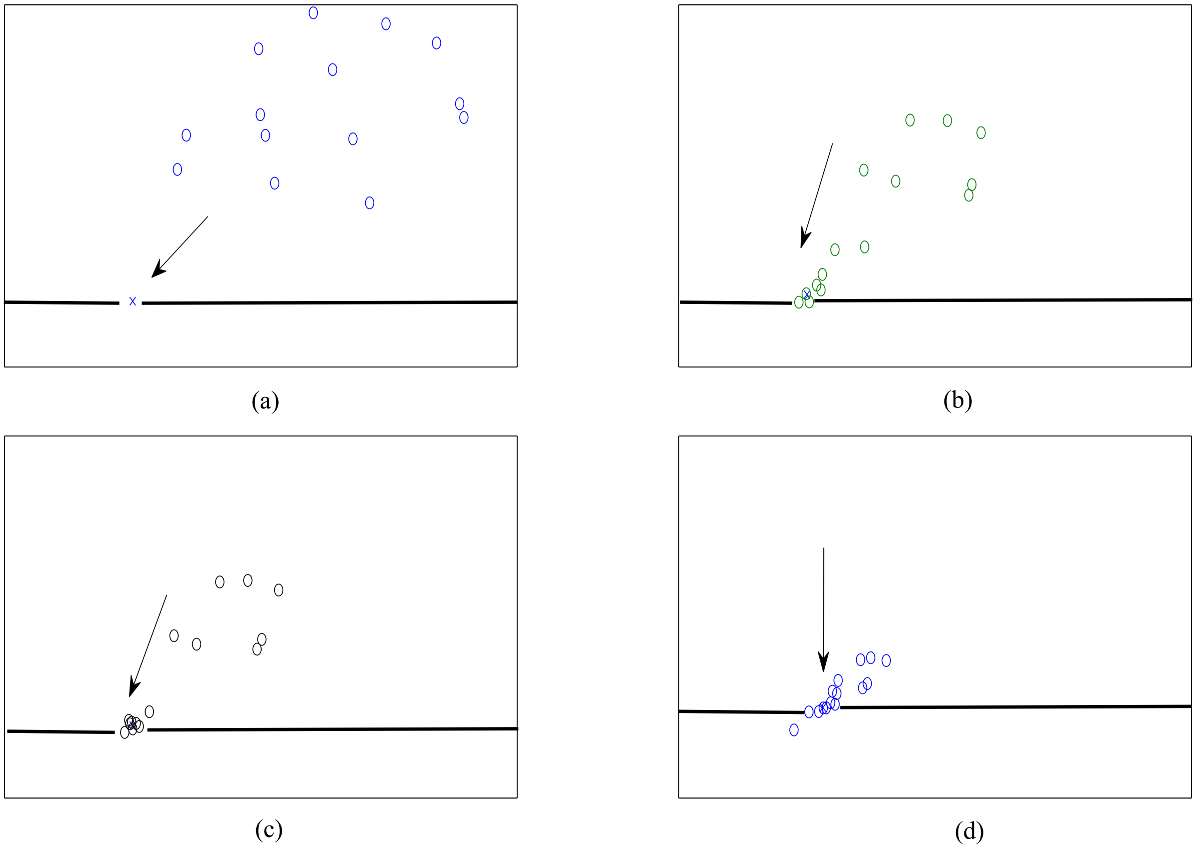

Fig. 9. Evacuation of a crowd through a single exit: $n = 0.2$ and $C_{\text{evac}} = 1.5$. (a) First instant. (b) Second instant. (c) Third instant. (d) Fourth instant.

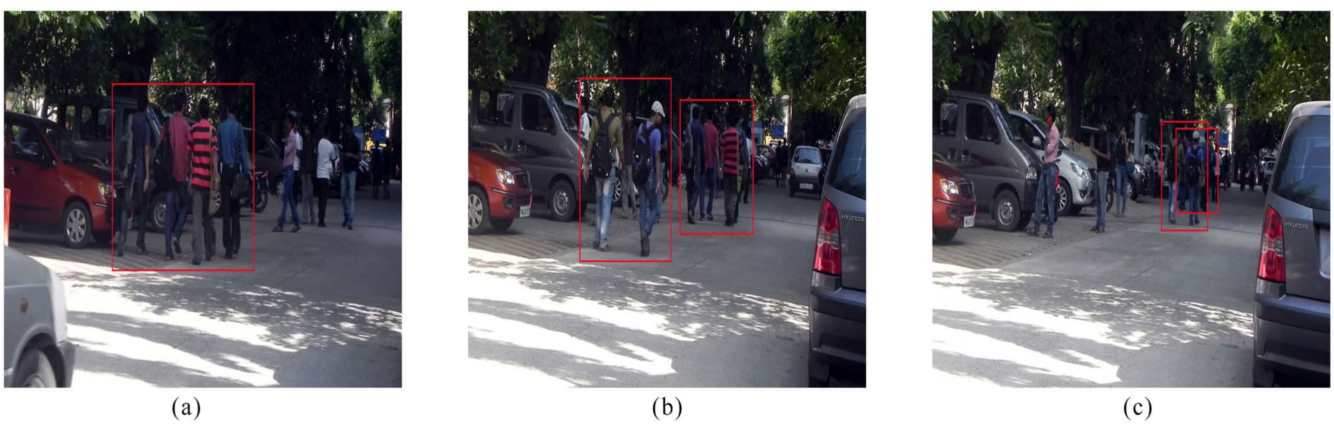

Fig. 10. Still images from video depicting pedestrian groups—two groups demarcated. (a) First instant. (b) Second instant. (c) Third instant.

VI. Conclusion

The ideas that we have proposed in this paper seek to provide a definite mathematical form to the various kinds of attractive and repulsive forces that are at work among the members of a crowd in motion in a 2-D space. We have sought to provide a fundamental justification behind the phenomena of convergence of velocities of the agents as well as "evacuation" and "panic" in a crowd system. That the crowd system forms a stable dynamics has been demonstrated via the Lyapunov energy function technique using the variable gradient method. We have justified the veracity of our claims by showing the working of the same principles in the real world, demonstrating the mechanics of group formation and portraying evacuation situations.

The novelty of the dynamics lies in the approach we have adopted to incorporate the basic features in crowd behavior. We resorted to the attractant-repellent profile even within a group so that the concepts of radius of comfort and distance for maximum attraction could find place. We used a potential function similar to the Lennard-Jones potential that suited the requirements of modeling. For the panic situations, this paper introduces the concept of degree of panic and we provide examples of situations where the degree of panic is indeed different. The rush towards the evacuation point is not uniform in all situations, but is specific to the scene. All these make contributions to the existing literature and attempt to extend the state-of-the-art in this domain of research.

Future work will focus on incorporating efficient evacuation design in modern architectures of public places. This will include a study of the crowd patterns in the specific site, its relevant simulation using our dynamics and subsequent decision making. The dynamics can find applications in other fields as well. One possibility is its use as a flocking dynamics to mimic the social behavior of birds.

Acknowledgment

The authors would like to thank the contributions of Dr. S. Das, Assistant Professor, Electronics and Communication Sciences Unit, Indian Statistical Institute, for his insightful discussions and guidelines on the research presented in this paper.

References

[1] E. Acar *et al.*, "Human action recognition using Lagrangian descriptors," in *Proc. IEEE 14th Int. Workshop Multimedia Signal Process. (MMSP)*, Banff, AB, Canada, 2012, pp. 360–365.

[2] A. R. Yu, B. B. Thompson, and R. J. Marks, II, "Competitive evolution of tactical multiswarm dynamics," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 43, no. 3, pp. 563–569, May 2013.

[3] S. Ali and M. Shah, "Floor fields for tracking in high density crowd scenes," in *Proc. 10th Eur. Conf. Comput. Vis.*, Marseille, France, 2008, pp. 1–14.

[4] S. Ali and M. Shah, "A Lagrangian particle dynamics approach for crowd flow segmentation and stability analysis," in *Proc. IEEE Conf. Comput. Vis. Pattern Recognit. (CVPR)*, Minneapolis, MN, USA, 2007, pp. 1–6.

[5] G. Antonini, M. Bierlaire, and M. Weber, "Discrete choice models of pedestrian walking behavior," *Transp. Res. B Met.*, vol. 40, no. 8, pp. 667–687, 2006.

[6] G. Antonini, S. Martinez, M. Bierlaire, and J. Thiran, "Behavioral priors for detection and tracking of pedestrians in video sequences," *Int. J. Comput. Vis.*, vol. 69, no. 2, pp. 159–180, 2006.

[7] N. Ashford, M. O'Leary, and P. D. McGinity, "Stochastic modelling of passenger and baggage flows through an airport terminal," *Traffic Eng. Control*, vol. 17, no. 5, pp. 207–210, 1976.

[8] N. Bellomo, B. Piccoli, and A. Tosin, "Modeling crowd dynamics from a complex system viewpoint," *Math. Models Methods Appl. Sci.*, vol. 22, Aug. 2012, Art. ID 123004.

[9] N. Bellomo, A. Bellouquid, and D. Knopoff, "From the micro-scale to collective crowd dynamics," *SIAM Multiscale Model. Simulat.*, vol. 11, no. 3, pp. 943–963, 2013.

[10] U. Blanke, T. Franke, G. Tröster, and P. Lukowicz, "Capturing crowd dynamics at large scale events using participatory GPS-localization," in *Proc. IEEE 9th Int. Conf. Intell. Sens. Sens. Netw. Inf. Process.*, Singapore, 2014, pp. 1–7.

[11] V. Coscia and C. Canavesio, "First order macroscopic modelling of human crowd dynamics," *Math. Model. Method Appl. Sci.*, vol. 18, pp. 1217–1247, Aug. 2008.

[12] I. Couzin and J. Krause, "Self-organization and collective behavior in vertebrates," *Adv. Study Behav.*, vol. 32, pp. 1–75, Jan. 2003.

[13] P. Degond, C. Appert-Rolland, J. Pettré, and G. Theraulaz, "Vision-based macroscopic pedestrian models," *Kinet. Relat. Models*, vol. 6, no. 4, pp. 809–839, 2013.

[14] N. Fridman, G. A. Kaminka, and A. Zilka, "The impact of culture on crowd dynamics: An empirical approach," in *Proc. Int. Conf. Autonom. Agents Multi-Agent Syst. (AAMAS)*, Richland, SC, USA, 2013, pp. 143–150.

[15] D. Garbrecht, "Describing pedestrian and car trips by transition matrices," *Traffic Quart.*, vol. 27, no. 1, pp. 89–109, 1973.

[16] D. Helbing, "A fluid-dynamic model for the movement of pedestrians," *Complex Syst.*, vol. 6, pp. 391–415, Jun. 1998.

[17] D. Helbing, "A mathematical model for the behavior of pedestrians," *Behav. Sci.*, vol. 36, no. 4, pp. 298–310, 1991.

[18] D. Helbing, I. Farkas, and T. Viscek, "Simulating dynamical feature of escape panic," *Nature*, vol. 407, pp. 487–490, Sep. 2000.

[19] D. Helbing and A. Johansson, "Pedestrian, crowd and evacuation dynamics," in *Encyclopedia of Complexity and Systems Science*. New York, NY, USA: Springer, 2009, pp. 6476–6495.

[20] D. Helbing, A. Johansson, and H. Z. Al-Abideen, "The dynamics of crowd disasters: An empirical study," *Phys. Rev. E*, vol. 75, Apr. 2007, Art. ID 046109.

[21] D. Helbing, A. Johansson, J. Mathiesen, H. M. Jensen, and A. Hansen, "Analytical approach to continuous and intermittent bottleneck flows," *Phys. Rev. Lett.*, vol. 97, Oct. 2006, Art. ID 168001.

[22] D. Helbing and P. Molnár, "Social force model for pedestrian dynamics," *Phys. Rev. E*, vol. 51, pp. 4282–4286, May 1995.

[23] L. F. Henderson, "On the fluid mechanics of human crowd motion," *Transp. Res.*, vol. 8, no. 6, pp. 509–515, 1974.

[24] S. Hoogendoorn and P. H. L. Bovy, "Simulation of pedestrian flows by optimal control and differential games," *Optim. Control Appl. Methods*, vol. 24, no. 3, pp. 153–172, 2003.

[25] S. Hoogendoorn and W. Daamen, "Pedestrian behavior at bottlenecks," *Transp. Sci.*, vol. 39, no. 2, pp. 147–159, 2005.

[26] R. L. Hughes, "A continuum theory for the flow of pedestrians," *Transp. Res. B Method.*, vol. 36, no. 6, pp. 507–535, 2002.

[27] A. Johansson, D. Helbing, and P. Shukla, "Specification of a microspopic pedestrian model by evolutionary adjustment to video tracking data," *Adv. Complex Syst.*, vol. 9, no. 10, pp. 271–288, 2008.

[28] T. Kretz, A. Grünebohm, M. Kaufman, F. Mazur, and M. Schreckenberg, "Experimental study of pedestrian counterflow in a corridor," *J. Stat. Mech. Theory Exp.*, vol. 2006, no. 10, 2006, Art. ID 10001.

[29] T. Kretz, A. Grünebohm, and M. Schreckenberg, "Experimental study of pedestrian flow through a bottleneck," *J. Stat. Mech. Theory Exp.*, vol. 2006, no. 10, 2006, Art. ID 10014.

[30] A. Kuhn, T. Senst, I. Keller, T. Sikora, and H. Theisel, "A Lagrangian framework for video analytics," in *Proc. IEEE 14th Int. Workshop Multimedia Signal Process. (MMSP)*, Banff, AB, Canada, 2012, pp. 387–392.

[31] M. Malisoff and F. Mazenc, *Construction of Strict Lyapunov Functions*. London, U.K.: Springer, 2009.

[32] A. A. Masoud, "Decentralized, self-organizing, potential field-based control for individually-motivated, mobile agents in a cluttered environment: A vector-harmonic potential field approach," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 37, no. 3, pp. 372–390, May 2007.

[33] B. Maury, A. Roudneff-Chupin, and F. Santambrogio, "A macroscopic crowd motion model of gradient flow type," *Math. Models Methods Appl. Sci.*, vol. 20, no. 10, pp. 1787–1821, 2010.

[34] B. Maury, A. Roudneff-Chupin, and F. Santambrogio, "Handling congestion in crowd motion modeling," *Netw. Heter. Media*, vol. 6, no. 3, pp. 485–519, 2011.

[35] R. Mehran and M. Shah, "Abnormal crowd behavior detection using social force model," in *Proc. IEEE Conf. Comput. Vis. Pattern Recognit. (CVPR)*, Miami, FL, USA, 2009, pp. 935–942.

[36] M. Moussaïd *et al.*, "Experimental study of the behavioural mechanisms underlying self-organization in human crowds," *Proc. Royal Soc. B Biological Sci.*, vol. 276, pp. 2755–2762, Jun. 2009.

[37] M. Moussaïd, N. Perozo, S. Garnier, D. Helbing, and G. Theraulaz, "The walking behaviour of pedestrian social groups and its impact on crowd dynamic," *PLoS ONE*, vol. 5, no. 4, 2010, Art. ID e10047.

[38] S. Al-Nasur and P. Kachroo, "A microscopic-to-macroscopic crowd dynamic model," in *Proc. IEEE Conf. Intell. Transp. Syst.*, Toronto, ON, Canada, 2006, pp. 606–611.

[39] V. Rabaud and S. Belongie, "Counting crowded moving objects," in *Proc. IEEE Comput. Soc. Conf. Comput. Vis. Pattern Recognit. (CVPR)*, vol. 1. New York, NY, USA, 2006, pp. 705–711.

[40] A. Schadschneider and A. Seyfried, "Empirical results for pedestrian dynamics and their implications for modeling," *Netw. Heter. Media*, vol. 6, no. 3, pp. 545–560, 2011.

[41] P. Scovanner and M. F. Tappen, "Learning pedestrian dynamics from the real world," in *Proc. IEEE 12th Int. Conf. Comput. Vis. (ICCV)*, Kyoto, Japan, 2009, pp. 381–388.

[42] H. Singh *et al.*, "Modelling subgroup behaviour in crowd dynamics DEM simulation," *Appl. Math. Model.*, vol. 33, no. 12, pp. 4408–4423, 2009.

[43] S. H. Strogatz, *Nonlinear Dynamics and Chaos*. Cambridge, MA, USA: Perseus, 1994.

[44] D. Sumpter, "The principles of collective animal behaviour," *Philosoph. Trans. R. Soc. B Biol. Sci.*, vol. 361, no. 1465, pp. 5–22, 2006.

[45] A. Treuille, S. Cooper, and Z. Popović, "Continuum crowds," in *Proc. ACM SIGGRAPH*, Boston, MA, USA, 2006, pp. 1160–1168.

[46] J. Was, "Cellular automata model of pedestrian dynamics for normal and evacuation conditions," in *Proc. 5th IEEE Conf. Intell. Syst. Design Appl. (ISDA)*, Warsaw, Poland, 2005, pp. 145–159.

[47] S. Wu, E. M. Brian, and M. Shah, "Chaotic invariants of Lagrangian particle trajectories for anomaly detection in crowded scenes," in *Proc. IEEE Conf. Comput. Vis. Pattern Recognit. (CVPR)*, San Francisco, CA, USA, 2010, pp. 2054–2060.

[48] E. Winston, R. J. Marks, II, B. B. Thompson, and A. Yu, "Evolutionary inversion of swarm emergence using disjunctive combs control," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 43, no. 5, pp. 1063–1076, Sep. 2013.

[49] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, "Determining Lyapunov exponents from a time series," *Phys. D Nonlin. Phenomena*, vol. 16, no. 3, pp. 285–317, 1985.

[50] D. B. Yang, H. H. Gonzalez-Banos, and L. J. Guibas, "Counting people in crowds with a real-time network of simple image sensors," in *Proc. 9th IEEE Int. Conf. Comput. Vis. (ICCV)*, vol. 1. Nice, France, 2003, pp. 122–129.

[51] W. Yu and A. Johansson, "Modeling crowd turbulence by many-particle simulations," *Phys. Rev. E*, vol. 76, Oct. 2007, Art. ID 046105.

[52] S. J. Yuhaski, Jr., and J. M. M. Smith, "Modelling circulation systems in buildings using state dependent queueing models," *Queueing Syst.*, vol. 4, no. 4, pp. 319–338, 1989.

[53] H. H. Zhang, "A non-equilibrium traffic model devoid of gas-like behaviour," *Transp. Res. B*, vol. 36, no. 3, pp. 275–290, 2002.

Sudipto Mukherjee is currently pursuing the B.E. degree from the Department of Electronics and Telecommunication Engineering, Jadavpur University, Kolkata, India.

His current research interests include multiagent systems, vision-based control, machine learning, and artificial intelligence. He has published research articles in peer-reviewed journals and international conference proceedings.

Debdipta Goswami was born in West Bengal, India, in 1993. He is currently pursuing the B.E. degree from the Department of Electronics and Telecommunication Engineering, Jadavpur University, Kolkata, India.

His current research interests include nonlinear systems, swarm intelligence, and multiagent behavior and automatic control. He has published research articles in peer-reviewed journals and international conference proceedings.

Sarthak Chatterjee was born in Kolkata, India, in 1992. He is currently pursuing the B.E. degree from the Department of Electronics and Telecommunication Engineering, Jadavpur University, Kolkata, India.

His current research interests include nonlinear dynamics and chaos theory, evolutionary computation, pattern recognition and image processing, and genetic programming. He has published research articles in peer-reviewed journals and international conference proceedings.