SOLUTIONS TO THE 2022 MIT INTEGRATION BEE FINALS

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Abstract. The MIT Integration Bee (which started at MIT in 1981) is the first of many integral calculus competitions that are held at educational institutions all over the world. Interested participants first take a qualifying exam, after which the successful candidates are drawn up against each other, tournament style, in brackets. Generally, the knockout rounds involve participants solving integrals in front of a chalkboard, often with a time limit. The champion gets the title of "Grand Integrator", along with prizes in the form of cash, vouchers, or books.

The 2022 MIT Integration Bee was held Thursday, January 20, 2022. This article demonstrates solutions, both original and curated, to the problems proposed in the Finals of the 2022 MIT Integration Bee.

Problem 1

The integral in question is

$$I = \int \sqrt{(\sin(20x) + 3\sin(21x) + \sin(22x))^2 + (\cos(20x) + 3\cos(21x) + \cos(22x))^2}dx.$$

We start with the complex number $z = e^{ix}$. Then

$$\begin{aligned} z^{20} + 3z^{21} + z^{22} &= \cos(20x) + i\sin(20x) + 3\cos(21x) + 3i\sin(21x) \\ &\quad + \cos(22x) + i\sin(22x) \\ &= (\cos(20x) + 3\cos(21x) + \cos(22x)) \\ &\quad + i(\sin(20x) + 3\sin(21x) + \sin(22x)), \end{aligned}$$

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Date: July 9, 2023.

and it is immediately obvious that the integrand is exactly equal to $|z^{20} + 3z^{21} + z^{22}|$. But

$$\begin{aligned}|z^{20} + 3z^{21} + z^{22}| &= |z^{21}(z + z^{-1} + 3)| \\ &= |z|^{21}|z + z^{-1} + 3| \\ &= |z + z^{-1} + 3| && \text{(since } |z| = 1) \\ &= |2\cos(x) + 3| && \text{(since } z + z^{-1} = 2\cos(x)) \\ &= 2\cos(x) + 3.\end{aligned}$$

Hence,

$$I = \int (2\cos(x) + 3)\, dx = \boxed{2\sin(x) + 3x + C}.$$

Problem 2

The second problem asks us to evaluate

$$J = \int_0^\infty \frac{e^{-2x}\sin(3x)}{x}\, dx.$$

Solution 1 We can neatly solve this by parametrizing the integral and then differentiating under the integral sign. Let

$$I(t) = \int_0^\infty \frac{e^{-2x}\sin(tx)}{x}\, dx.$$

Then, computing the derivative and then evaluating the resulting integral by parts, we get,

$$\begin{aligned}I'(t) &= \int_0^\infty \frac{\partial}{\partial t}\frac{e^{-2x}\sin(tx)}{x}\, dx \\ &= \int_0^\infty e^{-2x}\cos(tx)\, dx \\ &= \cos(tx)\frac{e^{-2x}}{-2}\Bigg|_0^\infty - \frac{t}{2}\int_0^\infty e^{-2x}\sin(tx)\, dx \\ &= \frac{1}{2} - \frac{t}{2}\left[\sin(tx)\frac{e^{-2x}}{-2}\Bigg|_0^\infty + \frac{t}{2}\int_0^\infty e^{-2x}\cos(tx)\, dx\right] \\ &= \frac{1}{2} - \frac{t^2}{4}I'(t).\end{aligned}$$

Thus,

$$I'(t) = \frac{2}{t^2+4}.$$

Note that $I(0) = 0$. Thus, integrating both sides from 0 to t yields

$$I(t) - I(0) = I(t) = \int_0^t \frac{2}{z^2+4}\,dz = \arctan\left(\frac{t}{2}\right).$$

Our desired integral is

$$J = I(3) = \boxed{\arctan\left(\frac{3}{2}\right)}.$$

Solution 2 An alternate solution of this problem is using Laplace transforms. We know that if $f(x)$ and $F(s)$ are Laplace transform pairs of each other, then,

$$\mathcal{L}\left\{\frac{f(x)}{x}\right\} = \int_s^\infty F(\sigma)\,d\sigma.$$

Accordingly, if $f(x) = \sin(3x)$, then

$$\mathcal{L}\left\{\frac{f(x)}{x}\right\}(s) = \int_0^\infty e^{-sx}\frac{f(x)}{x}\,dx = \int_0^\infty e^{-sx}\frac{\sin(3x)}{x}\,dx,$$

and so,

$$J = \mathcal{L}\left\{\frac{f(x)}{x}\right\}(2).$$

But $F(\sigma)$, the Laplace transform of a sine, $\sin(kx)$, is

$$F(\sigma) = \frac{k}{\sigma^2+k^2}.$$

So,

$$J = \mathcal{L}\left\{\frac{f(x)}{x}\right\}(2) = \int_{s=2}^\infty \frac{3}{\sigma^2+3^2}d\sigma = \arctan\left(\frac{\sigma}{3}\right)\Bigg|_2^\infty = \frac{\pi}{2} - \arctan\left(\frac{2}{3}\right) = \boxed{\arctan\left(\frac{3}{2}\right)}.$$

Problem 3

Problem 3 is a beastly-looking integral and asks us to find the value of

$$I = \int_0^{2\pi} \cos(2022x)\frac{\sin(10050x)}{\sin(50x)}\frac{\sin(10251x)}{\sin(51x)}\,dx.$$

The following beautiful solution is due to the user `chronondecay` on AoPS, who originally proposed this problem. We use complex numbers. Define $z = e^{ix}$. Then, the integrand is equal to

$$\begin{aligned}\operatorname{Re}&\left(z^{-2022}\cdot\frac{z^{10050}-z^{-10050}}{z^{50}-z^{-50}}\cdot\frac{z^{10251}-z^{-10251}}{z^{51}-z^{-51}}\right)\\ &=\operatorname{Re}\left(z^{-2022}\cdot\sum_{a=-100}^{100}z^{100a}\cdot\sum_{b=-100}^{100}z^{102b}\right)\\ &=\operatorname{Re}\left(\sum_{|a|,|b|\leqslant 100}z^{100a+102b-2022}\right).\end{aligned}$$

This is because we can sum up the two geometric series as follows:

$$\sum_{a=-100}^{100}z^{100a}=\frac{z^{-10000}(z^{100\times 201}-1)}{z^{100}-1}=\frac{z^{10100}-z^{-10000}}{z^{100}-1}=\frac{z^{10050}-z^{-10050}}{z^{50}-z^{-50}},$$

and,

$$\sum_{b=-100}^{100}z^{102b}=\frac{z^{-10200}(z^{102\times 201}-1)}{z^{102}-1}=\frac{z^{10302}-z^{-10200}}{z^{102}-1}=\frac{z^{10251}-z^{-10251}}{z^{51}-z^{-51}}.$$

If we integrate this, we get a value of 2π if and only if the exponent of z is 0. This is because $\int_0^{2\pi} z^n\,dx = 2\pi$ when $n = 0$ and $\int_0^{2\pi} z^n\,dx = 0$ when $n \neq 0$. So what we are actually looking for when trying to find the indices that yield a non-zero integral are the integer solutions of the equation $100a+102b-2022 = 0$ in the intervals $-100 \leqslant a, b \leqslant 100$. The integer solutions of this equation are of the form $a = 9 + 51n, b = 11 - 50n$, where $n \in \mathbb{Z}$. Our only solutions in the intervals of interest, $-100 \leqslant a, b \leqslant 100$, are $(a, b) = (9, 11), (a, b) = (60, -39)$, and $(a, b) = (-42, 61)$. Each of these 3 ordered pairs yields 2π as the integral. Hence, our integral is

$$I=\int_0^{2\pi}\cos(2022x)\frac{\sin(10050x)}{\sin(50x)}\frac{\sin(10251x)}{\sin(51x)}\,dx=3\times 2\pi=\boxed{6\pi}.$$

Problem 4

Problem 4 asks us to evaluate

$$I=\int_0^1 x^{1/3}(1-x)^{2/3}\,dx.$$

This is just the fundamental definition of the beta function

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

Hence,

$$\begin{aligned} I = B\left(\frac{4}{3}, \frac{5}{3}\right) = \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{4}{3}+\frac{5}{3}\right)} = \frac{\Gamma\left(1+\frac{1}{3}\right)\Gamma\left(1+\frac{2}{3}\right)}{\Gamma(3)} &= \frac{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{2!} \\ &= \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) \\ &= \frac{1}{9}\frac{\pi}{\sin(\pi/3)} \\ &= \boxed{\frac{2\pi}{9\sqrt{3}}}. \end{aligned}$$

The relationships we have used are that

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

the recurrence relation for the gamma function

$$\Gamma(x+1) = x\Gamma(x),$$

and the Euler reflection formula for the gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \qquad x \notin \mathbb{Z}.$$

Problem 5

Problem 5 asks us to find the value of

$$C = \left\lfloor \log_{10} \int_{2022}^{\infty} 10^{-x^3}\,dx \right\rfloor.$$

Start with the embedded integral

$$I = \int_{2022}^{\infty} 10^{-x^3}\,dx.$$

We use the substitution $t = x^3 \ln(10)$. Then, $dt = 3x^2 \ln(10)\,dx$. Our integral then becomes

$$I = \int_{2022^3 \ln(10)}^{\infty} 10^{-t/\ln(10)} \frac{dt}{3\ln(10)\left(\frac{t}{\ln(10)}\right)^{2/3}} = \frac{1}{3(\ln(10))^{1/3}} \int_{2022^3 \ln(10)}^{\infty} t^{-2/3} e^{-t} dt.$$

We can now enlist the help of the upper incomplete gamma function

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt,$$

and, therefore,

$$I = \frac{1}{3(\ln(10))^{1/3}}\Gamma\left(\frac{1}{3}, 2022^3\ln(10)\right).$$

For large values of x (which is certainly true in this case), we use the asymptotic expansion of the upper incomplete gamma function [1]

$$\Gamma(s,x) \sim x^{s-1}e^{-x}\left[1 + \frac{s-1}{x} + \frac{(s-1)(s-2)}{x^2} + \dots\right].$$

We can get an upper bound for the integral by truncating the series above to the first term as

$$\begin{aligned} I < C_1 = \frac{(2022^3\ln(10))^{-2/3}e^{-2022^3\ln(10)}}{3(\ln(10))^{1/3}} &= \frac{10^{-2022^3}}{2022^2 \times 3 \times \ln(10)} \\ &< \frac{10^{-2022^3}}{2000^2 \times 3 \times 2} \\ &= \frac{10^{-2022^3-6}}{24} \\ &< \frac{10^{-2022^3-6}}{10} \\ &= 10^{-2022^3-7}. \end{aligned}$$

Additionally, a lower bound can be obtained as follows from [2]:

$$\Gamma(s,x) > \frac{e^{-x}}{2s}\left[(x+2)^s - x^s\right].$$

Plugging in the required quantities, we have

$$\begin{aligned} I > C_2 &= \frac{e^{-2022^3\ln(10)}}{3 \times (\ln(10))^{1/3} \times \frac{2}{3}}\left[(2022^3\ln(10)+2)^{1/3} - (2022^3\ln(10))^{1/3}\right] \\ &= \frac{10^{-2022^3}}{2 \times (\ln(10))^{1/3}} \times 9.35 \times 10^{-8} \\ &= 3.54 \times 10^{-2022^3-8} \\ &> 10^{-2022^3-8}. \end{aligned}$$

Now, since

$$10^{-2022^3-8} < I < 10^{-2022^3-7},$$

we have that

$$-2022^3 - 8 < \log_{10} I < -2022^3 - 7.$$

Therefore,

$$C = \left\lfloor \log_{10} \int_{2022}^{\infty} 10^{-x^3}\, dx \right\rfloor = \boxed{-2022^3 - 8}.$$

References

[1] Abramowitz, Milton, Irene A. Stegun, and Robert H. Romer. "Handbook of mathematical functions with formulas, graphs, and mathematical tables." (1988): 958-958.

[2] Gautschi, Walter. "Some elementary inequalities relating to the gamma and incomplete gamma function." *J. Math. Phys.* 38.1 (1959): 77-81.