SOLUTIONS TO THE 2023 MIT INTEGRATION BEE FINALS

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ABSTRACT. The MIT Integration Bee (which started at MIT in 1981) is the first of many integral calculus competitions that are held at educational institutions all over the world. Interested participants first take a qualifying exam, after which the successful candidates are drawn up against each other, tournament style, in brackets. Generally, the knockout rounds involve participants solving integrals in front of a chalkboard, often with a time limit. The champion gets the title of "Grand Integrator", along with prizes in the form of cash, vouchers, or books.

The 2023 MIT Integration Bee was held Thursday, January 26, 2023. This article demonstrates solutions, both original and curated, to the problems proposed in the Finals of the 2023 MIT Integration Bee.

Problem 1

Problem 1 of the 2023 integration bee finals asks us to evaluate

$$I = \int_0^{\pi/2} \frac{\sqrt[3]{\tan x}}{(\sin x + \cos x)^2} dx.$$

We first factor out $\cos x$ from the denominator. What we have then is

$$I = \int_0^{\pi/2} \frac{\sqrt[3]{\tan x}}{(\tan x + 1)^2} \sec^2 x \, dx.$$

This looks neat and we can substitute $\tan x = z$, which means that $dz = \sec^2 x dx$. Our integral reduces to

$$I = \int_0^{\infty} \frac{z^{1/3}}{(1+z)^2} \, dz.$$

The above integral can be tackled with a judicious choice of the many forms of the integral representation of the beta function. In our case, that choice turns

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out to be

$$B(m,n) = \int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\,dz.$$

One comparison with our integral tells us that we need to solve $m-1 = 1/3$ and $m+n = 2$ and so, $m = 4/3$ and $n = 2 - 4/3 = 2/3$. Therefore,

$$I = B\left(\frac{4}{3}, \frac{2}{3}\right).$$

We now need two properties. The first is the relationship of the beta function to the gamma function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

and the second is the fundamental recurrence relation for the gamma function,

$$\Gamma(y+1) = y\Gamma(y)$$

on an appropriate domain. We can then write

$$\begin{aligned} I = B\left(\frac{4}{3}, \frac{2}{3}\right) = \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}+\frac{2}{3}\right)} = \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(2)} &= \frac{\Gamma\left(1+\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(2)} \\ &= \frac{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(2)}. \end{aligned}$$

For positive integers n, $\Gamma(n) = (n-1)!$ and so, $\Gamma(2) = 1! = 1$. Therefore,

$$I = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right).$$

We can then use Euler's reflection formula for the gamma function, which states that for non-integral y,

$$\Gamma(1-y)\Gamma(y) = \frac{\pi}{\sin(\pi y)},$$

and can finally wrap things up by writing

$$I = \frac{1}{3}\cdot\frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{3}\cdot\frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}} = \boxed{\frac{2\sqrt{3}\pi}{9}}.$$

Problem 2

The second problem asks us to annihilate the following formidable opponent:

$$I = \int_0^\pi \left(\frac{\sin(2x)\sin(3x)\sin(5x)\sin(30x)}{\sin(x)\sin(6x)\sin(10x)\sin(15x)} \right)^2 \, dx.$$

Lots of angle/double-angle combinations are apparent at a first glance. So let us use the double-angle formula for the sine, $\sin(2x) = 2\sin x \cos x$ on all of the double-angle arguments of the sine in the integrand. Our integral then reduces to

$$I = \int_0^\pi \left(\frac{\cos(x)}{\cos(3x)} \cdot \frac{\cos(15x)}{\cos(5x)} \right)^2 \, dx.$$

We then use the triple-angle formula for the cosine, $\cos(3x) = 4\cos^3 x - 3\cos x$. Dividing throughout by $\cos x$ yields

$$\frac{\cos(3x)}{\cos x} = 4\cos^2 x - 3.$$

Therefore,

$$I = \int_0^\pi \left(\frac{4\cos^2(5x) - 3}{4\cos^2(x) - 3} \right)^2 \, dx.$$

Next, we use the double-angle formula for the cosine $\cos(2x) = 2\cos^2 x - 1$. Hence,

$$I = \int_0^\pi \left(\frac{2(1+\cos(10x)) - 3}{2(1+\cos(2x)) - 3} \right)^2 \, dx = \int_0^\pi \left(\frac{2\cos(10x) - 1}{2\cos(2x) - 1} \right)^2 \, dx.$$

We call the quantity in the parentheses z. Utilizing the identity $e^{ix} + e^{-ix} = 2\cos x$, and defining $y \coloneqq e^{ix}$, we have that

$$z = \frac{2\cos(10x) - 1}{2\cos(2x) - 1} = \frac{y^{10} - 1 + y^{-10}}{y^2 - 1 + y^{-2}} = \frac{y^{16} + y^{14} - y^{10} - y^8 - y^6 + y^2 + 1}{y^8}.$$

Let us further simplify the above. We get

$$z = y^8 + y^6 - y^2 - 1 - y^{-2} + y^{-6} + y^{-8},$$

and, after a little bit of rearrangement

$$z = (y^8 + y^{-8}) + (y^6 + y^{-6}) - (y^2 + y^{-2}) - 1,$$

which means

$$z = 2\cos(8x) + 2\cos(6x) - 2\cos(2x) - 1.$$

Our integrand is the square of this expression, which is

$$\begin{aligned} z^2 &= (4\cos^2(2x) + 4\cos^2(6x) + 4\cos^2(8x) + 1) \\ &\quad - 8\cos(6x)\cos(2x) - 8\cos(8x)\cos(2x) + 8\cos(6x)\cos(8x) \\ &\quad + 4\cos(2x) - 4\cos(6x) - 4\cos(8x). \end{aligned}$$

The above expression can be further simplified using the formulae $\cos(2x) = 2\cos^2 x - 1$ as well as the product-to-sum formula for the cosine $2\cos a \cos b = \cos(a+b) + \cos(a-b)$ to yield as our integrand

$$\begin{aligned} z^2 &= 2\cos(16x) + 4\cos(14x) + 2\cos(12x) - 4\cos(10x) \\ &\quad - 8\cos(8x) - 8\cos(6x) - 2\cos(4x) + 8\cos(2x) + 7. \end{aligned}$$

Integrating each of the cosine terms between 0 and π yields zero. We are thus left with

$$I = \int_0^{\pi} 7\,dx = \boxed{7\pi}.$$

Problem 3

On then to Problem 3, which looks like this:

$$I = \int_{-1/2}^{1/2} \sqrt{x^2 + 1 + \sqrt{x^4 + x^2 + 1}}\,dx.$$

We try and express the integrand as a sum of two radicals, i.e.,

$$\sqrt{x^2 + 1 + \sqrt{x^4 + x^2 + 1}} = \sqrt{a} + \sqrt{b}. \tag{†}$$

Squaring both sides,

$$x^2 + 1 + \sqrt{x^4 + x^2 + 1} = a + b + 2\sqrt{ab}.$$

By inspection, one solution of this equation is

$$a + b = x^2 + 1, \qquad 2\sqrt{ab} = \sqrt{x^4 + x^2 + 1}.$$

From the second equation, we have

$$4ab = x^4 + x^2 + 1.$$

Using $(a-b)^2 = (a+b)^2 - 4ab$, we have

$$(a-b)^2 = (x^2+1)^2 - (x^4 + x^2 + 1) = x^2.$$

Hence,

$$a + b = x^2 + 1, \qquad a - b = \pm x.$$

Thus,

$$a = \frac{x^2 \pm x + 1}{2}, \qquad b = \frac{x^2 \mp x + 1}{2}.$$

Utilizing the symmetry on the right hand side of (†), we can argue that both the positive as well as the negative square root lead to the same pair of solutions for a and b. Without loss of generality, we take the positive square root. We can then immediately write

$$I = \frac{1}{\sqrt{2}} \left(\underbrace{\int_{-1/2}^{1/2} \sqrt{x^2 + x + 1}\, dx}_{I_1} + \underbrace{\int_{-1/2}^{1/2} \sqrt{x^2 - x + 1}\, dx}_{I_2} \right).$$

We then attempt to separately evaluate the integrals I_1 and I_2. Accordingly,

$$I_1 = \int_{-1/2}^{1/2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx.$$

We substitute

$$x + \frac{1}{2} = \frac{\sqrt{3}}{2} \sinh u,$$

and, so,

$$dx = \frac{\sqrt{3}}{2} \cosh u\, du.$$

Thus,

$$I_1 = \int_0^{\sinh^{-1} \frac{2}{\sqrt{3}}} \left(\frac{\sqrt{3}}{2} \cosh u\right) \cdot \frac{\sqrt{3}}{2} \cosh u\, du = \frac{3}{4} \int_0^{\sinh^{-1} \frac{2}{\sqrt{3}}} \cosh^2 u\, du.$$

We can similarly write

$$I_2 = \int_{-1/2}^{1/2} \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx,$$

and carrying out an almost identical copy of the substitution performed above yields

$$I_2 = \frac{3}{4} \int_{-\sinh^{-1} \frac{2}{\sqrt{3}}}^{0} \cosh^2 u\, du.$$

Thus,

$$\begin{aligned}\sqrt{2}I = I_1 + I_2 &= \frac{3}{4}\left(\int_0^{\sinh^{-1}\frac{2}{\sqrt{3}}} \cosh^2 u \, du + \int_{-\sinh^{-1}\frac{2}{\sqrt{3}}}^{0} \cosh^2 u \, du\right) \\ &= \frac{3}{8}\int_{-\sinh^{-1}\frac{2}{\sqrt{3}}}^{\sinh^{-1}\frac{2}{\sqrt{3}}} 2\cosh^2 u \, du \\ &= \frac{3}{8}\int_{-\sinh^{-1}\frac{2}{\sqrt{3}}}^{\sinh^{-1}\frac{2}{\sqrt{3}}} (1+\cosh(2u)) \, du\end{aligned}$$

and therefore,

$$I = \frac{3}{8\sqrt{2}}\left(u + \frac{\sinh(2u)}{2}\right)\Bigg|_{-\sinh^{-1}\frac{2}{\sqrt{3}}}^{\sinh^{-1}\frac{2}{\sqrt{3}}}.$$

Performing the algebra, the integral works out to

$$I = \boxed{\frac{\sqrt{7}}{2\sqrt{2}} + \frac{3}{4\sqrt{2}}\ln\left(\frac{\sqrt{7}+2}{\sqrt{3}}\right)}.$$

Problem 4

Problem 4 of the bee asks us to find the value of

$$C = \left\lfloor 10^{20}\int_2^{\infty} \frac{x^9}{x^{20} - 48x^{10} + 575} \, dx \right\rfloor.$$

In many ways, this was the simplest problem of the bee, with only the floor function proving to induce a bit of complexity but also making it an absolutely delightful peach of a problem! The integral itself is not too difficult.

We start with the integral inside the floor function,

$$I = \int_2^{\infty} \frac{x^9}{x^{20} - 48x^{10} + 575} \, dx.$$

Let $z = x^{10} - 24$ and so $dz = 10x^9 \, dx$. We can write

$$I = \frac{1}{10}\int_2^{\infty} \frac{10x^9}{(x^{10}-24)^2 - 1} \, dx = \frac{1}{10}\int_{2^{10}-24}^{\infty} \frac{dz}{z^2-1} = \frac{1}{10}\int_{1000}^{\infty} \frac{dz}{(z+1)(z-1)}.$$

This screams "partial fractions", therefore,

$$I = \frac{1}{20}\int_{1000}^{\infty}\left(\frac{1}{z-1} - \frac{1}{z+1}\right) dz,$$

and,

$$I = \frac{1}{20}\ln\left(\frac{z-1}{z+1}\right)\Bigg|_{1000}^{\infty} = \frac{1}{20}\left(\ln(1) - \ln\left(\frac{999}{1001}\right)\right) = \frac{1}{20}\ln\left(\frac{1001}{999}\right).$$

How do we deal with the floor function? Let us look at the Taylor series expansion for the logarithm function

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

and

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots.$$

Therefore, subtracting the second equation from the first,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right),$$

with all of the above formulae being valid for $|x| < 1$. Now,

$$\begin{aligned} C = \left\lfloor \frac{10^{20}}{20}\ln\left(\frac{1001}{999}\right)\right\rfloor &= \left\lfloor \frac{10^{20}}{20}\ln\left(\frac{1+10^{-3}}{1-10^{-3}}\right)\right\rfloor \\ &= \left\lfloor \frac{10^{20}}{20}\cdot 2\left(10^{-3} + \frac{10^{-9}}{3} + \frac{10^{-15}}{5} + \frac{10^{-21}}{7} + \dots\right)\right\rfloor \\ &= \left\lfloor 10^{19}\left(10^{-3} + \frac{10^{-9}}{3} + \frac{10^{-15}}{5} + \frac{10^{-21}}{7} + \dots\right)\right\rfloor \\ &= \left\lfloor 10^{16} + \frac{10^{10}}{3} + \frac{10^{4}}{5} + (\text{terms smaller than } 1)\right\rfloor. \end{aligned}$$

Both 10^{16} as well as $10^4/5$ are integers and $10^{10} \equiv 1 \pmod 3$. Hence, our elusive constant is

$$C = \boxed{10^{16} + \frac{10^{10}-1}{3} + \frac{10^4}{5} = 10000003333335333},$$

and we are done!

Problem 5

The last problem of the competition is an absolute beauty:

$$I = \int_0^1 \left(\sum_{n=1}^{\infty} \frac{\lfloor 2^n x \rfloor}{3^n} \right)^2 dx.$$

In what follows, I will present a solution due to the user `chronondecay` on the Art of Problem Solving (AoPS) community, who was the original writer of this problem. What follows below is `chronondecay`'s beautiful solution.

Assume that

$$f(x) = \sum_{n=1}^{\infty} \frac{\lfloor 2^n x \rfloor}{3^n},$$

which is the sum inside the parentheses. The first key realization is to spot the $2^n x$ term in the numerator and realize that we can smell a variable being multiplied by progressive powers of 2. This is a good motivator for us to start looking at the binary representation of the variable of integration x.

Suppose our variable of integration $x \in [0, 1)$ has a binary representation of $x = (0.a_1 a_2 a_3 \dots)_2$. So,

$$x = (0.a_1 a_2 a_3 \dots)_2 = \sum_{k \geqslant 1} a_k 2^{-k},$$

where $a_k \in \{0, 1\}$ and the sequence $\{a_k\}_{k \geqslant 1}$ is not eventually all recurring 1s. So then, in base 2,

$$2x = (a_1 \cdot a_2 a_3 a_4 \dots)_2, \qquad \lfloor 2x \rfloor = (a_1)_2,$$

$$2^2 x = (a_1 a_2 \cdot a_3 a_4 \dots)_2, \qquad \lfloor 2^2 x \rfloor = (a_1 a_2)_2,$$

$$2^3 x = (a_1 a_2 a_3 \cdot a_4 \dots)_2, \qquad \lfloor 2^3 x \rfloor = (a_1 a_2 a_3)_2,$$

and so on. Thus,

$$\begin{aligned} f(x) &= \frac{(a_1)_2}{3} + \frac{(a_1 a_2)_2}{3^2} + \frac{(a_1 a_2 a_3)_2}{3^3} + \dots \\ &= \frac{2^0 a_1}{3} + \frac{2^1 a_1 + 2^0 a_2}{3^2} + \frac{2^2 a_1 + 2^1 a_2 + 2^0 a_3}{3^3} + \dots . \end{aligned}$$

So now,

$$f(x) = a_1 \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \dots \right) + a_2 \left(\frac{1}{3^2} + \frac{2}{3^3} + \frac{2^2}{3^4} + \dots \right) + \dots,$$

or,

$$f(x) = \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots\right)\left(\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \dots\right).$$

The infinite geometric series in the first pair of parentheses sums to 3. Therefore,

$$f(x) = 3\sum_{k=1}^{\infty} a_k 3^{-k},$$

and thus, the original integral is

$$I = \int_0^1 f(x)^2\,dx.$$

We then finish this integral off by interpreting it probabilistically. The integral is simply the expected value of $f(x)^2$ when x is chosen uniformly randomly in the interval $[0, 1)$. But, based on the definition of $f(x)$, this is also equivalent to choosing the binary digits $a_k, k \geqslant 1$ uniformly from $\{0, 1\}$, independently of each other (see Appendix for a discussion of this problem). Thus,

$$I = \int_0^1 f(x)^2\,dx = \mathbb{E}[f(x)^2] = \mathbb{E}\left[\left(3\sum_{k=1}^{\infty} a_k 3^{-k}\right)^2\right] = 9\mathbb{E}\left[\left(\sum_{k=1}^{\infty} a_k 3^{-k}\right)^2\right].$$

This can be written as

$$I = 9\mathbb{E}\left[\left(\sum_{k=1}^{\infty} a_k^2 9^{-k} + \sum_{k\neq l} a_k a_l 3^{-k-l}\right)\right],$$

using the independence of the a_k-s. This means (since $\mathbb{E}[a_k] = 1/2\ \forall\, k$),

$$I = 9\left(\frac{1}{4}\sum_{k=1}^{\infty} 9^{-k} + \frac{1}{4}\left(\sum_{k=1}^{\infty} 3^{-k}\right)^2\right) = \frac{9}{4}\left(\frac{1/9}{1-1/9} + \left(\frac{1/3}{1-1/3}\right)^2\right),$$

and so,

$$I = \frac{9}{4}\left(\frac{1}{8} + \left(\frac{1}{2}\right)^2\right) = \boxed{\frac{27}{32}}.$$

Appendix

There is a beautiful result in probability theory that states that if $X_1, X_2, \ldots$ are i.i.d. Bernoulli($\frac{1}{2}$) random variables, then

$$Z = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$$

is uniformly distributed on $[0, 1]$. A little bit of investigation reveals that identifying the general family of the distribution function of

$$Z = \sum_{n=1}^{\infty} b^{-n} X_n,$$

where $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$ and $b > 1$, is a problem on which copious amounts of research has been done. The cumulative distribution function F_Z of Z is known as an *infinite Bernoulli convolution*. See, for example, [1], and the references therein.

Additionally, for any real number $y \in [0, 1]$, let its binary expansion be

$$y = \sum_{k=1}^{\infty} b_k 2^{-k}$$

for $b_k \in \{0, 1\}$. Then, if we consider the function

$$C_z(y) = \sum_{k=1}^{\infty} b_k z^k,$$

then the inverse of $x = 2C_{1/3}(y)$ is the Cantor function [2].

References

[1] Li, Deli, Yongcheng Qi, and Andrew Rosalsky. "On the Set of Limit Points of Normed Sums of Geometrically Weighted I.I.D. Bounded Random Variables." *Stochastic Analysis and Applications* 28.1 (2009): 86-102.

[2] Dovgoshey, Oleksiy, Olli Martio, Vladimir Ryazanov, and Matti Vuorinen. "The Cantor function." *Expositiones Mathematicae* 24.1 (2006): 1-37.