# SOLUTIONS TO THE 2024 MIT INTEGRATION BEE PRELIMS

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$\boxed{1}$

$$I = \int_{2023}^{2025} 2024\,dx = 2024x\Big|_{2023}^{2025} = 2024 \times 2 = \boxed{4048}.$$

$\boxed{2}$

$$I = \int \frac{(x-1)^{\log(x+1)}}{(x+1)^{\log(x-1)}}\,dx.$$

Assume $z = \frac{(x-1)^{\log(x+1)}}{(x+1)^{\log(x-1)}}$, so

$$\log z = \log(x+1)\log(x-1) - \log(x-1)\log(x+1) = 0.$$

Hence, $z = 1$, and

$$I = \int dx = \boxed{x + C}.$$

$\boxed{3}$

$$I = \int (x\log x + 2x)\,dx = \log x \int x\,dx - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx + x^2 = \boxed{\frac{x^2\log x}{2} + \frac{3x^2}{4} + C}.$$

(The first integral has been evaluated by parts).

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$$I = \int \frac{dx}{x(\log x + 2)} = \boxed{\log\ \left|\log x + 2\right| + C}.$$

($u$-substitution with $u = \log x + 2$).

$\boxed{5}$

$$I = \int_0^{2\pi} \arccos(\sin x)\,dx = x\arccos(\sin x)\Big|_0^{2\pi} - \int_0^{2\pi} -\frac{\cos x}{\sqrt{1-\sin^2 x}}\cdot x\,dx = \boxed{\pi^2}.$$

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$$I=\int \frac{\cos x+\cot x+\csc x+1}{\sin x+\tan x+\sec x+1}\,dx=\int \frac{\frac{\sin x\cos x+\cos x+1+\sin x}{\sin x}}{\frac{\sin x\cos x+\sin x+1+\cos x}{\cos x}}\,dx=\int \cot x\,dx=\boxed{\log|\sin x|+C}.$$

$\boxed{7}$

$$\begin{aligned}I=\int \frac{x^{2024}-1}{x^{506}-1}\,dx&=\frac{(x^{1012}+1)(x^{506}+1)(x^{506}-1)}{x^{506}-1}\,dx\\&=\int (x^{1518}+x^{1012}+x^{506}+1)\,dx\\&=\boxed{\frac{x^{1519}}{1519}+\frac{x^{1013}}{1013}+\frac{x^{507}}{507}+x+C}.\end{aligned}$$

$\boxed{8}$

$$I=\int_{-1}^{1}(5x^3-3x)^2\,dx=\int_{-1}^{1}(25x^6-30x^4+9x^2)\,dx=\frac{25x^7}{7}-6x^5+3x^3\Big|_{-1}^{1}=\boxed{\frac{8}{7}}.$$

$\boxed{9}$

$$I=\int_0^{2\pi}(\sin x+\cos x)^{11}\,dx.$$

If we do manage to expand the binomial and integrate term by term, we will notice that the result of the integration will only be linear combinations of powers of sines and cosines with their arguments being odd multiples of $x$. Since the sine and the cosine are both $2\pi$-periodic, the integrals will all evaluate to zero. Hence,

$$I=\int_0^{2\pi}(\sin x+\cos x)^{11}\,dx=\boxed{0}.$$

$\boxed{10}$

$$\begin{aligned}I=\int_0^{2\pi}(\sinh x+\cosh x)^{11}\,dx&=\int_0^{2\pi}\left(\frac{e^x-e^{-x}+e^x+e^{-x}}{2}\right)^{11}\,dx\\&=\int_0^{2\pi}e^{11x}\,dx=\frac{e^{11x}}{11}\Big|_0^{2\pi}=\boxed{\frac{e^{22\pi}-1}{11}}.\end{aligned}$$

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$$I = \int \csc^2 x \tan^{2024} x \, dx = \int \frac{1}{\sin^2 x} \cdot \frac{\sin^{2024} x}{\cos^{2024} x} \, dx$$
$$= \int \frac{1}{\cos^2 x} \cdot \frac{\sin^{2022} x}{\cos^{2022} x} \, dx$$
$$= \int \sec^2 x \tan^{2022} x \, dx$$
$$= \boxed{\frac{\tan^{2023} x}{2023} + C}.$$

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$$I = \int \cos^x x (\log \cos x - x \tan x) \, dx.$$

Assume that $u = \cos^x x$. So, $\log u = x \log \cos x$, therefore,

$$\frac{du}{u} = \left( \log \cos x - x \cdot \frac{1}{\cos x} \cdot \sin x \right) dx = (\log \cos x - x \tan x) \, dx.$$

So,

$$I = \int du = u + C = \boxed{\cos^x x + C}.$$

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$$I = \int_{-\infty}^{\infty} e^{-(x-2024)^2/4} \, dx.$$

We can perform the substitution $u = (x - 2024)/2$. The integral becomes

$$I = 2 \int_{-\infty}^{\infty} e^{-u^2} \, du = \boxed{2\sqrt{\pi}}.$$

(We have used the well-known result that the Gaussian integral evaluates to $\sqrt{\pi}$ over the real line).

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$$I = \int_{1/e}^{e} \left(1 - \frac{1}{x^2}\right) e^{e^{x+1/x}} dx.$$

Use the substitution $x = 1/u$. We then get

$$I = -\int_{e}^{1/e} (1 - u^2) e^{e^{u+1/u}} \frac{du}{u^2} = -\int_{1/e}^{e} \left(1 - \frac{1}{u^2}\right) e^{e^{u+1/u}} du = -I.$$

Thus,

$$\int_{1/e}^{e} \left(1 - \frac{1}{x^2}\right) e^{e^{x+1/x}} dx = \boxed{0}.$$

$\boxed{15}$

$$I = \int (x + 1 - e^{-x}) e^{xe^x} \, dx.$$

We try the substitution $u = e^{x(e^x-1)}$. Then

$$du = e^{x(e^x-1)}(e^x + xe^x - 1) \, dx = e^{xe^x}(x + 1 - e^{-x}) \, dx.$$

Thus,

$$I = \int du = u + C = \boxed{e^{x(e^x-1)} + C}.$$

$\boxed{16}$

$$\begin{aligned} I = \int \left(\frac{\arctan x}{1-x^2} + \frac{\operatorname{arctanh} x}{1+x^2}\right) dx &= \int (\arctan x) d\,(\operatorname{arctanh} x) + (\operatorname{arctanh} x) d\,(\arctan x) \\ &= \int d(\arctan x \operatorname{arctanh} x) \\ &= \boxed{\arctan x \operatorname{arctanh} x + C}. \end{aligned}$$

$\boxed{17}$

$$\begin{aligned} I = \int \left(\sum_{k=0}^{\infty} \sin\left(\frac{k\pi}{2}\right) x^k\right) dx &= \int \left(x - x^3 + x^5 - x^7 + \dots\right) dx \\ &= \int \frac{x \, dx}{1+x^2} = \boxed{\frac{1}{2}\log(1+x^2) + C}. \end{aligned}$$

$\boxed{18}$

$$\begin{aligned} I = \int_0^1 \left(\sum_{n=0}^{2024} x^{2^{n-1012}}\right) dx &= \int_0^1 \left(x^{2^{-1012}} + x^{2^{-1011}} + x^{2^{-1010}} + \dots + x + \dots x^{2^{1010}} + x^{2^{1011}} + x^{2^{1012}}\right) dx \\ &= \frac{1}{1+2^{-1012}} + \frac{1}{1+2^{-1011}} + \dots + \frac{1}{2} + \dots + \frac{1}{1+2^{1011}} + \frac{1}{1+2^{1012}} \\ &= \frac{2^{1012}}{1+2^{1012}} + \frac{2^{1011}}{1+2^{1011}} + \dots + \frac{1}{2} + \dots + \frac{1}{1+2^{1011}} + \frac{1}{1+2^{1012}}. \end{aligned}$$

The above sum has 2025 terms. We note that the first and the last term sum to 1, and so does the second term from the beginning and the second term from the end, and so on. There are 1012 total pairs like these, each summing to 1. Thus,

$$I = 1012 + \frac{1}{2} = \boxed{\frac{2025}{2}}.$$

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$$I = \int \frac{x^4}{3 - 6x + 6x^2 - 4x^3 + 2x^4}\,dx.$$

We can perform long division of the integrand to get

$$\frac{x^4}{3 - 6x + 6x^2 - 4x^3 + 2x^4} = \frac{1}{2} + \frac{2x^3 - 3x^2 + 3x - \frac{3}{2}}{3 - 6x + 6x^2 - 4x^3 + 2x^4} = \frac{1}{2} + \frac{1}{4}\left(\frac{8x^3 - 12x^2 + 12x - 6}{3 - 6x + 6x^2 - 4x^3 + 2x^4}\right).$$

Then,

$$I = \boxed{\frac{x}{2} + \frac{1}{4}\log\left|(3 - 6x + 6x^2 - 4x^3 + 2x^4)\right| + C}.$$

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$$I = \int_1^3 \frac{x + \frac{x+\cdots}{1+\cdots}}{1 + \frac{x+\cdots}{1+\cdots}}\,dx.$$

Quite clearly, if the integrand is $f$, we need to solve

$$f = \frac{x + f}{1 + f} \implies f^2 + f = x + f \implies f = \sqrt{x},$$

since the integrand is obviously positive. So,

$$I = \int_1^3 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^3 = \boxed{2\sqrt{3} - \frac{2}{3}}.$$