MIT Integration Bee 2024 Quarterfinals

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1 QF1: Problem 1

$$I_1 = \int \ln x \left(\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x \right) \mathrm{d}x.$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{e}\right)^x = \ln x \cdot \left(\frac{x}{e}\right)^x \tag{1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e}{x}\right)^x = -\ln x \cdot \left(\frac{e}{x}\right)^x \tag{2}$$

Subtracting (2) from (1) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{e}\right)^x - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e}{x}\right)^x = \ln x \cdot \left(\frac{x}{e}\right)^x + \ln x \cdot \left(\frac{e}{x}\right)^x.$$

Therefore, our integrand is an exact differential. Thus,

$$I_1 = \int \ln x \left(\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x \right) \mathrm{d}x = \boxed{\left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + C}.$$

2 QF1: Problem 2

$$I_2 = \int_0^\infty \frac{\sin^3 x}{x} \, \mathrm{d}x.$$

We use the triple angle formula for the sine, $\sin 3x = 3\sin x - 4\sin^3 x$. Thus,

$$\begin{aligned} I_2 = \frac{3}{4}\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x - \frac{1}{4}\int_0^\infty \frac{\sin 3x}{x}\,\mathrm{d}x &= \frac{3}{4}\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x - \frac{3}{4}\int_0^\infty \frac{\sin 3x}{3x}\,\mathrm{d}x \\ &= \frac{3}{4}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{3}\cdot\frac{\pi}{2} \\ &= \boxed{\frac{\pi}{4}}. \end{aligned}$$

We have used the standard result that the Dirichlet integral $(\sin x)/x$ integrated over the positive real line is $\pi/2$.

Note: An alternative solution of this problem is using Lobachevsky's Dirichlet integral formula, which states that if $f(x)$ is a continuous function satisfying $f(x+\pi) = f(\pi - x) = f(x)$ for $x \in [0, \infty)$, then

$$\int_0^\infty \frac{\sin^2 x}{x^2} f(x)\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{x} f(x)\,\mathrm{d}x = \int_0^{\pi/2} f(x)\,\mathrm{d}x.$$

By a straightforward calculation,

$$I_2 = \int_0^\infty \frac{\sin^3 x}{x}\,\mathrm{d}x = \int_0^{\pi/2} \sin^2 x\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi/2} (1-\cos 2x)\,\mathrm{d}x = \boxed{\frac{\pi}{4}}.$$

3 QF1: Problem 3

$$I_3 = \int \begin{vmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{vmatrix}\,\mathrm{d}x.$$

The matrix in question is a tridiagonal matrix. Tridiagonal matrices have a beautiful property that the determinant of a tridiagonal matrix (sometimes also called a *continuant*) can be computed from a three-term recurrence relation. We can set the recurrence relation for this case up by noting that if

$$D_n = \begin{vmatrix} x & 1 & & & \\ 1 & x & 1 & & \\ & & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & x \end{vmatrix},$$

then $D_n = xD_{n-1} - 1\cdot 1\cdot D_{n-2} = xD_{n-1} - D_{n-2}$. Clearly, $D_1 = x$ and $D_2 = x^2 - 1$. Then,

$$D_3 = xD_2 - D_1 = x^3 - 2x,$$

$$D_4 = xD_3 - D_2 = x^4 - 3x^2 + 1,$$

and

$$D_5 = xD_4 - D_3 = x^5 - 4x^3 + 3x.$$

Hence,

$$I_3 = \int \begin{vmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{vmatrix}\,\mathrm{d}x = \int (x^5 - 4x^3 + 3x)\,\mathrm{d}x = \boxed{\frac{x^6}{6} - x^4 + \frac{3x^2}{2} + C}.$$

4 QF2: Problem 1

$$I_4 = \lim_{n\to\infty}\left(\int_0^1 \sum_{k=1}^n \frac{(kx)^4}{n^5}\,\mathrm{d}x\right).$$

Inside the parentheses (ignoring a few technicalities which allow us to do this), we can swap the order of integration and summation to yield

$$I_4 = \lim_{n\to\infty}\sum_{k=1}^n \frac{k^4}{n^5}\int_0^1 x^4\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^n \frac{k^4}{5n^5}.$$

To calculate the sum, we use Faulhaber's formula, which expresses the sum of the $p-$th powers of the first n positive integers, and is given by

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{r=0}^{p}\binom{p+1}{r} B_r n^{p-r+1},$$

where the B_r-s are the Bernoulli numbers. For the case when $p=4$, we have

$$I_4 = \lim_{n\to\infty} \frac{1}{5n^5}\cdot\frac{1}{5}\left(n^5+\frac{5}{2}n^4+\frac{5}{3}n^3-\frac{1}{6}n\right) = \boxed{\frac{1}{25}}.$$

5 QF2: Problem 2

$$I_5 = \int_0^1 \frac{\ln(1+x^2+x^3+x^4+x^5+x^6+x^7+x^9)}{x}\,\mathrm{d}x.$$

The argument of the logarithm factorizes as $(1+x^2)(1+x^3)(1+x^4)$. Then,

$$I_5 = \int_0^1 \frac{\ln(1+x^2)}{x}\,\mathrm{d}x + \int_0^1 \frac{\ln(1+x^3)}{x}\,\mathrm{d}x + \int_0^1 \frac{\ln(1+x^4)}{x}\,\mathrm{d}x.$$

Hence, what we need is a general representation of

$$\int_0^1 \frac{\ln(1+x^n)}{x}\,\mathrm{d}x.$$

We can write (since $|x|<1$)

$$\begin{aligned}\int_0^1 \frac{\ln(1+x^n)}{x}\,\mathrm{d}x &= \int_0^1 \left(x^{n-1}-\frac{x^{2n-1}}{2}+\frac{x^{3n-1}}{3}-\frac{x^{4n-1}}{4}+\dots\right)\mathrm{d}x\\ &= \left(\frac{x^n}{n}-\frac{x^{2n}}{2^2 n}+\frac{x^{3n}}{3^2 n}-\dots\right)\Bigg|_0^1\\ &= \frac{1}{n}\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\dots\right)\\ &= \frac{\pi^2}{12n}, n>0.\end{aligned}$$

We have used the fact that the final infinite series being summed is the Dirichlet eta function evaluated at 2, $\eta(2)=\pi^2/12$. Thus,

$$I_5 = \frac{\pi^2}{12\cdot 2}+\frac{\pi^2}{12\cdot 3}+\frac{\pi^2}{12\cdot 4} = \boxed{\frac{13\pi^2}{144}}.$$

6 QF2: Problem 3

$$I_6 = \int_0^1 (1-\sqrt[2024]{x})^{2024}\,\mathrm{d}x.$$

We use the substitution $x=u^{2024}$, then $\mathrm{d}x = 2024u^{2023}\,\mathrm{d}u$. Thus,

$$\begin{aligned}I_6 &= 2024\int_0^1 u^{2023}(1-u)^{2024}\,\mathrm{d}u = 2024\mathrm{B}(2024,2025)\\ &= 2024\frac{\Gamma(2024)\Gamma(2025)}{\Gamma(4049)} = 2024\frac{2023!\times 2024!}{4048!} = \frac{(2024!)^2}{4048!} = \boxed{\frac{1}{\binom{4048}{2024}}}.\end{aligned}$$

7 QF3: Problem 1

$$I_7 = \int_0^{2\pi} \Big|\{\lfloor \sin x \rfloor, \lfloor \cos x \rfloor, \lfloor \tan x \rfloor, \lfloor \cot x \rfloor\}\Big| \, \mathrm{d}x.$$

We investigate the behavior of the integral by splitting the interval from $[0, \pi/2), [\pi/2, \pi), [\pi, 3\pi/2)$, and $[3\pi/2, 2\pi)$. Clearly, we can write

$$x \in [0, \pi/2) \implies \Big|\{\lfloor \sin x \rfloor, \lfloor \cos x \rfloor, \lfloor \tan x \rfloor, \lfloor \cot x \rfloor\}\Big| = \Big|\{0, \text{some positive number}\}\Big| = 2,$$

$$x \in [\pi/2, \pi) \implies \Big|\{\lfloor \sin x \rfloor, \lfloor \cos x \rfloor, \lfloor \tan x \rfloor, \lfloor \cot x \rfloor\}\Big| = \Big|\{-1, 0, \text{some negative number}\}\Big| = 3,$$

$$x \in [\pi, 3\pi/2) \implies \Big|\{\lfloor \sin x \rfloor, \lfloor \cos x \rfloor, \lfloor \tan x \rfloor, \lfloor \cot x \rfloor\}\Big| = \Big|\{-1, 0, \text{some positive number}\}\Big| = 3,$$

$$x \in [3\pi/2, 2\pi) \implies \Big|\{\lfloor \sin x \rfloor, \lfloor \cos x \rfloor, \lfloor \tan x \rfloor, \lfloor \cot x \rfloor\}\Big| = \Big|\{-1, 0, \text{some negative number}\}\Big| = 3.$$

Thus,

$$I_7 = \int_0^{\pi/2} 2 \, \mathrm{d}x + \int_{\pi/2}^{\pi} 3 \, \mathrm{d}x + \int_{\pi}^{3\pi/2} 3 \, \mathrm{d}x + \int_{3\pi/2}^{2\pi} 3 \, \mathrm{d}x = \boxed{\frac{11\pi}{2}}.$$

8 QF3: Problem 2

$$I_8 = \int_0^{\infty} \frac{\mathrm{d}x}{(x+1)(\ln^2 x + \pi^2)}.$$

Assume $\ln x = u \implies x = e^u$. Thus,

$$I_8 = \int_{-\infty}^{\infty} \frac{e^u \, \mathrm{d}u}{(e^u+1)(u^2+\pi^2)} \overset{u \mapsto -u}{=} \int_{-\infty}^{\infty} \frac{e^{-u} \, \mathrm{d}u}{(e^{-u}+1)(u^2+\pi^2)} = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{(e^u+1)(u^2+\pi^2)}.$$

Adding the first integral above with the third, we get

$$2I_8 = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{u^2+\pi^2} = \frac{1}{\pi} \cdot \pi \implies I_8 = \boxed{\frac{1}{2}}.$$

9 QF3: Problem 3

$$I_9 = \lim_{n\to\infty} \frac{1}{n} \int_0^n \max\left(\{x\}, \{\sqrt{2}x\}, \{\sqrt{3}x\}\right) \, \mathrm{d}x.$$

As far as problems go, this is perhaps the most cleverly constructed problem of the lot and is a direct consequence of Weyl's equidistribution theorem, which states that the fractional parts of the multiples of an irrational number are uniformly distributed. Additionally, we also require the result that if we have n random variables $X_1, X_2, \ldots, X_n$, all distributed uniformly, $X_i \sim \text{Uniform}(0,1)$, then

$$\mathbb{E}\left[\max\{X_1, X_2, \ldots, X_n\}\right] = \frac{n}{n+1}.$$

A proof can be found in `https://jamesmccammon.com/2017/02/18/finding-the-expected-value-of-the-maximum-of-n-random-variables`. Here we seek to find the expected value of 3 such random variables, so,

$$I_9 = \lim_{n\to\infty} \frac{1}{n} \int_0^n \max\left(\{x\}, \{\sqrt{2}x\}, \{\sqrt{3}x\}\right) \mathrm{d}x = \frac{3}{3+1} = \boxed{\frac{3}{4}}.$$

10 QF4: Problem 1

$$I_{10} = \int \frac{e^{2x}}{(1-e^x)^{2024}} \mathrm{d}x.$$

Assume $1 - e^x = u$. Then,

$$\begin{aligned} I_{10} &= -\int \frac{(1-u)\,\mathrm{d}u}{u^{2024}} = \int \frac{\mathrm{d}u}{u^{2023}} - \int \frac{\mathrm{d}u}{u^{2024}} = \frac{1}{2023u^{2023}} - \frac{1}{2022u^{2022}} + C \\ &= \boxed{\frac{1}{2023(1-e^x)^{2023}} - \frac{1}{2022(1-e^x)^{2022}} + C}. \end{aligned}$$

11 QF4: Problem 2

$$I_{11} = \lim_{n\to\infty} \log_n \left(\int_0^1 (1-x^3)^n \,\mathrm{d}x \right).$$

For the inner integral, assume $x^3 = u \implies 3x^2\,\mathrm{d}x = \mathrm{d}u$. Thus,

$$\int_0^1 (1-x^3)^n \,\mathrm{d}x = \frac{1}{3}\int_0^1 u^{-2/3}(1-u)^n \,\mathrm{d}u = \frac{1}{3}\mathrm{B}(1/3, n+1) = \frac{1}{3}\frac{\Gamma(1/3)\Gamma(n+1)}{\Gamma(n+4/3)}.$$

We invoke the asymptotic series for $\ln\Gamma(z)$, which is nothing but a variation of the Stirling approximation,

$$\ln\Gamma(z) \sim \left(z - \frac{1}{2}\right)\ln z \implies \log_z \Gamma(z) \sim \left(z - \frac{1}{2}\right).$$

So,

$$I_{11} = \lim_{n\to\infty} \log_n \left(\frac{\Gamma(n+1)}{\Gamma(n+4/3)} = \right) = n + 1 - \frac{1}{2} - n - \frac{4}{3} + \frac{1}{2} = \boxed{-\frac{1}{3}}.$$

12 QF4: Problem 3

$$I_{12} = \int \frac{\sin x}{1+\sin x} \cdot \frac{\cos x}{1+\cos x} \,\mathrm{d}x.$$

We can use the tangent half-angle substitution $t = \tan(x/2)$. We can write

$$I_{12} = \int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} \cdot \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{2t}{(t+1)^2} \cdot \frac{(1+t)(1-t)}{2} \cdot \frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{2t(1-t)}{(t+1)(t^2+1)} \,\mathrm{d}t.$$

A partial fraction expansion of the above yields

$$\begin{aligned} I_{12} = \int \left(\frac{2}{t^2+1} - \frac{2}{t+1} \right) \mathrm{d}t &= 2\arctan t - 2\ln|t+1| + C' = x - 2\ln|1 + \tan(x/2)| + C' \\ &= \boxed{x + \ln\left| \left(\frac{1+\cos x}{1+\sin x} \right) \right| + C}. \end{aligned}$$

13 Tiebreaker Problem 1

$$I_{13} = \int_0^{2024} x^{2024} \log_{2024} x \, \mathrm{d}x.$$

We can transform first into the natural logarithm and then use $u = x/2024$ to yield

$$\begin{aligned} I_{13} = \frac{2024^{2025}}{\ln 2024} \int_0^1 u^{2024} \ln(2024u) \, \mathrm{d}u &= 2024^{2025} \int_0^1 u^{2024} \, \mathrm{d}u + \frac{2024^{2025}}{\ln 2024} \int_0^1 u^{2024} \ln u \, \mathrm{d}u \\ &= \boxed{\frac{2024^{2025}}{2025} - \frac{2024^{2025}}{2025^2 \ln 2024}}. \end{aligned}$$

Note that we have used the result

$$\int_0^1 x^n \ln x \, \mathrm{d}x = -\frac{1}{(n+1)^2}, \quad \mathrm{Re}(x) > -1.$$

14 Tiebreaker Problem 2

$$I_{14} = \lim_{t\to\infty} \int_0^2 \left(x^{-2024t} \prod_{n=1}^{2024} \sin(nx^t) \right) \mathrm{d}x.$$

We can write

$$I_{14} = \lim_{t\to\infty} \int_0^1 \left(x^{-2024t} \prod_{n=1}^{2024} \sin(nx^t) \right) \mathrm{d}x + \lim_{t\to\infty} \int_1^2 \left(x^{-2024t} \prod_{n=1}^{2024} \sin(nx^t) \right) \mathrm{d}x.$$

The quantity inside the parentheses is just

$$\frac{\sin x^t}{x^t} \cdot \frac{\sin 2x^t}{x^t} \cdot \frac{\sin 3x^t}{3x^t} \cdot \dots \cdot \frac{\sin 2024x^t}{x^t}.$$

When $x \in [1, 2)$, the integrand of the second integral above vanishes as $t \to \infty$. Also, when $x \in [0, 1)$, we can invoke the small-angle approximation for the sines in the first integrand since the arguments of the latter become smaller and smaller as $t \to \infty$. Thus,

$$I_{14} = \lim_{t\to\infty} \int_0^1 \frac{x^t}{x^t} \cdot \frac{2x^t}{x^t} \cdot \frac{3x^t}{3x^t} \cdot \dots \cdot \frac{2024x^t}{x^t} \, \mathrm{d}x = \boxed{2024!}.$$