MIT Integration Bee 2024 Regular Season

Sarthak Chatterjee

January 28, 2024

1 Problem 1

$$I_1 = \int_1^{2024} \lfloor \log_{43}(x) \rfloor \, \mathrm{d}x.$$

The only powers of 43 that lie in the domain of integration are $1, 43$, and 1849. Thus,

$$I_1 = 0\int_1^{43} \mathrm{d}x + 1\int_{43}^{1849} \mathrm{d}x + 2\int_{1849}^{2024} \mathrm{d}x = 0 \cdot (43-1) + 1 \cdot (1849-43) + 2 \cdot (2024-1849) = \boxed{2156}.$$

2 Problem 2

$$I_2 = \int \frac{\mathrm{d}x}{x^{2024} - x^{4047}}.$$

We can recast this as

$$I_2 = \int \frac{\mathrm{d}x}{x^{4047}(x^{-2023}-1)} = \int \frac{\mathrm{d}x}{x^{2023} \cdot x^{2024}(x^{-2023}-1)}.$$

Substitute $u = x^{-2023}$. Then, $\mathrm{d}u = -2023x^{-2024}\,\mathrm{d}x$. So,

$$I_2 = -\frac{1}{2023}\int \frac{u\,\mathrm{d}u}{u-1} = -\frac{1}{2023}\int \mathrm{d}u - \frac{1}{2023}\int \frac{\mathrm{d}u}{u-1} = -\frac{1}{2023}x^{-2023} - \frac{1}{2023}\ln(x^{-2023}-1) + C$$
$$= \boxed{\ln x - \frac{1}{2023}\ln(1-x^{2023}) - \frac{1}{2023}x^{-2023} + C}.$$

3 Problem 3

$$I_3 = \int_0^1 x^2(1-x)^{2024}\,\mathrm{d}x.$$

This is just the beta function $\mathrm{B}(m, n)$ with $m = 3$ and $n = 2025$. Hence,

$$I_3 = \mathrm{B}(m,n) = \frac{\Gamma(3)\Gamma(2025)}{\Gamma(2028)} = \frac{2! \times 2024!}{2027!} = \boxed{\frac{2}{2027 \times 2026 \times 2025}}.$$

4 Problem 4

$$I_4 = \int \frac{2023x+1}{x^2+2024} \mathrm{d}x.$$

Our approach will be to split the above into two elementary integrals.

$$\begin{aligned} I_4 &= \frac{2023}{2} \int \frac{2x}{x^2+2024}\, \mathrm{d}x + \int \frac{\mathrm{d}x}{x^2+2024} \\ &= \boxed{\frac{2023}{2} \ln(x^2+2024) + \frac{1}{\sqrt{2024}} \arctan\left(\frac{x}{\sqrt{2024}}\right) + C}. \end{aligned}$$

5 Problem 5

$$I_5 = \int_0^{\pi/2} \sec^2(x) e^{-\sec^2(x)}\, \mathrm{d}x.$$

We use the substitution $u = \tan x$ and, so, $\sec^2(x)\, \mathrm{d}x = \mathrm{d}u$.

$$I_5 = \int_0^\infty e^{-(1+u^2)}\, \mathrm{d}u = \frac{1}{e} \int_0^\infty e^{-u^2}\, \mathrm{d}u = \boxed{\frac{\sqrt{\pi}}{2e}}.$$

We have used the fact that the Gaussian integral evaluates to $\sqrt{\pi}/2$ over the positive real line.

6 Problem 6

$$I_6 = \int \cot x \cot 2x \mathrm{d}x = \int \frac{\cos x}{\sin x} \cdot \frac{\cos 2x}{\sin 2x} \mathrm{d}x = \int \frac{1-2\sin^2 x}{2\sin^2 x} \mathrm{d}x = \frac{1}{2} \int \csc^2 x \mathrm{d}x - \int \mathrm{d}x = \boxed{-x - \frac{\cot x}{2} + C}.$$

7 Problem 7

$$\begin{aligned} I_7 = \int \frac{\sinh^2 x}{\tanh 2x}\, \mathrm{d}x = \int \frac{\sinh^2 x(2\cosh^2 x - 1)}{2\sinh x \cosh x}\, \mathrm{d}x &= \frac{1}{2} \int \sinh 2x\, \mathrm{d}x - \frac{1}{2} \int \tanh x\, \mathrm{d}x \\ &= \boxed{\frac{1}{4} \cosh 2x - \frac{1}{2} \ln \cosh x + C}. \end{aligned}$$

8 Problem 8

$$I_8 = \int \arctan \sqrt{x}\, \mathrm{d}x.$$

Use the substitution $x = u^2$, therefore $\mathrm{d}x = 2u\, \mathrm{d}u$. Thus,

$$\begin{aligned} I_8 = \int 2u \arctan u\, \mathrm{d}u = 2 \arctan u \cdot \frac{u^2}{2} - 2 \int \frac{1}{u^2+1} \cdot \frac{u^2}{2}\, \mathrm{d}u &= u^2 \arctan u - \int \mathrm{d}u + \int \frac{\mathrm{d}u}{u^2+1} \\ &= x \arctan \sqrt{x} - \sqrt{x} + \arctan \sqrt{x} + C \\ &= \boxed{(x+1) \arctan \sqrt{x} - \sqrt{x} + C}. \end{aligned}$$

9 Problem 9

$$I_9 = \int_0^\infty \frac{x \ln x}{x^4+1}\,\mathrm{d}x.$$

Use the substitution $x = 1/u$, therefore $\mathrm{d}x = -1/u^2\,\mathrm{d}u$. Thus,

$$I_9 = -\int_\infty^0 \frac{\frac{1}{u}\ln(1/u)}{1+1/u^4}\cdot\frac{\mathrm{d}u}{u^2} = -\int_0^\infty \frac{u\ln u}{u^4+1}\,\mathrm{d}u = -I_9 \implies I_9 = \boxed{0}.$$

10 Problem 10

$$I_{10} = \int_0^{10} \lfloor x\lfloor x\rfloor\rfloor\,\mathrm{d}x.$$

We solve this by splitting the domain of integration. We will investigate the behavior of the integral when x is in $[0,1)$, $[1,2)$, $[2,3)$, and so on. This gives us

$$I_{10} = \int_0^1 \lfloor 0x\rfloor\,\mathrm{d}x + \int_1^2 \lfloor x\rfloor\,\mathrm{d}x + \int_2^3 \lfloor 2x\rfloor\,\mathrm{d}x + \int_3^4 \lfloor 3x\rfloor\,\mathrm{d}x + \cdots + \int_9^{10} \lfloor 9x\rfloor\,\mathrm{d}x.$$

Since the floor function has jump discontinuities at the integers, the first integral above will be zero, the second integral will be 1 (scaled by 1), the third integral will be the sum of two integrals each having value 1/2 (scaled by 4 and 5), the fourth integral will be the sum of three integrals each having value 1/3 (scaled by 9, 10, and 11), and so on. Thus,

$$\begin{aligned} I_{10} &= 0+1+\frac{4+5}{2}+\frac{9+10+11}{3}+\frac{16+\cdots+19}{4}+\frac{25+\cdots+29}{5}+\frac{36+\cdots+41}{6} \\ &+\frac{49+\cdots+55}{7}+\frac{64+\cdots+71}{8}+\frac{81+\cdots+89}{9} = \boxed{303}. \end{aligned}$$

11 Problem 11

$$I_{11} = \int_0^1 e^{-x}\sqrt{1+\cot^2(\arccos e^{-x})}\,\mathrm{d}x.$$

We can equivalently write

$$I_{11} = \int_0^1 e^{-x}\csc\arccos e^{-x}\,\mathrm{d}x.$$

We may verify using a simple calculation that

$$\csc\arccos e^{-x} = \frac{1}{\sqrt{1-e^{-2x}}},$$

over the domain of integration, and, therefore,

$$I_{11} = \int_0^1 \frac{e^{-x}\,\mathrm{d}x}{\sqrt{1-e^{-2x}}}.$$

Substitute $e^{-x} = u$. Thus,

$$I_{11} = -\int_1^{1/e} \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \int_{1/e}^1 \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \arcsin x\Big|_{1/e}^1 = \arcsin 1 - \arcsin(1/e) = \boxed{\frac{\pi}{2} - \arcsin(e^{-1})}.$$

12 Problem 12

$$I_{12} = \int_1^3 \frac{1 + \frac{1+\dots}{x+\dots}}{x + \frac{1+\dots}{x+\dots}}\,\mathrm{d}x.$$

If the integrand is f, then, clearly

$$f = \frac{1+f}{x+f} \implies fx + f^2 = 1 + f \implies f^2 + (x-1)f - 1 = 0 \implies f = \frac{1 - x + \sqrt{(x-1)^2+4}}{2},$$

because the integrand is positive over the domain of integration. Hence,

$$I_{12} = \frac{1}{2}\int_1^3 \mathrm{d}x - \frac{1}{2}\int_1^3 x\,\mathrm{d}x + \frac{1}{2}\int_1^3 \sqrt{(x-1)^2+2^2}\,\mathrm{d}x = 1 - 2 + \sqrt{2} + \sinh^{-1} 1 = \boxed{\sqrt{2} - 1 + \ln(1+\sqrt{2})}.$$

Note that the final integral can be tackled by either using the hyperbolic substitution $x - 1 = 2\sinh u$ or by using the standard integral

$$\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x + \sqrt{x^2+a^2}) + C.$$

13 Problem 13

$$\begin{aligned} I_{13} = \int_0^1 \frac{2x(1-x)^2}{1+x^2}\,\mathrm{d}x = \int_0^1 \frac{2x^3 - 4x^2 + 2x}{x^2+1}\,\mathrm{d}x = \int_0^1 2x\,\mathrm{d}x - \int \frac{4x^2}{x^2+1} \\ = 1 - 4\int_0^1 \mathrm{d}x + 4\int_0^1 \frac{\mathrm{d}x}{x^2+1} = \boxed{\pi - 3}. \end{aligned}$$

The integrand is positive everywhere in the domain of integration and we can use this fact to conclude that $\pi > 3$.

14 Problem 14

$$I_{14} = \int e^{e^x + 3x}\,\mathrm{d}x.$$

Substituting $e^x = u$, gives us

$$I_{14} = \int \frac{e^u u^3\,\mathrm{d}u}{u} = \int u^2 e^u\,\mathrm{d}u = e^u(u^2 - 2u + 2) + C = \boxed{e^{e^x}(e^{2x} - 2e^x + 2) + C}.$$

15 Problem 15

$$I_{15} = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} 2\left(1 - \frac{|x|}{\sqrt{3}}\right)\,\mathrm{d}x.$$

We can write

$$I_{15} = \int_{-\sqrt{3}/2}^{0} 2\left(1 + \frac{x}{\sqrt{3}}\right)\,\mathrm{d}x + \int_0^{\sqrt{3}/2} 2\left(1 - \frac{x}{\sqrt{3}}\right)\,\mathrm{d}x = \sqrt{3} + \sqrt{3} - \frac{3}{4\sqrt{3}} - \frac{3}{4\sqrt{3}} = \boxed{\frac{3\sqrt{3}}{2}}.$$

Alternatively, the integral is also double the area of the trapezoid formed by the lines $x = \sqrt{3}/2$, $x = 0$, $y = 0$, and $y = 2(1 - x/\sqrt{3})$, which evaluates to $3\sqrt{3}/2$.

16 Problem 16

$$I_{16} = \int \frac{\ln(1+x^2)}{x^2}\,\mathrm{d}x = \ln(1+x^2)\int \frac{\mathrm{d}x}{x^2} - \int \frac{2x}{1+x^2}\cdot\left(-\frac{1}{x}\right)\,\mathrm{d}x = \boxed{2\arctan x - \frac{\ln(1+x^2)}{x} + C}.$$

17 Problem 17

$$I_{17} = \int 2^x x^2\,\mathrm{d}x.$$

Assume $u = 2^x$, and, thus, $\mathrm{d}u = 2^x \ln 2\,\mathrm{d}x$. Then,

$$I_{17} = \frac{1}{(\ln 2)^3}\int (\ln u)^2\,\mathrm{d}u = \frac{u}{(\ln 2)^3}(\ln^2 u - 2\ln u + 2) + C = \boxed{\frac{2^x}{\ln^3 2}(x^2\ln^2 2 - 2x\ln 2 + 2) + C}.$$

18 Problem 18

$$I_{18} = \int_0^1 \sqrt{x^8 - x^6 + x^4}\cdot\sqrt{1+x^2}\,\mathrm{d}x.$$

Some algebra simplifies the integrand to $x^2\sqrt{x^6+1}$. From there, we can substitute $x^6 + 1 = u^2$. This leads us to,

$$I_{18} = \frac{1}{3}\int_1^{\sqrt{2}} \frac{u^2\,\mathrm{d}u}{\sqrt{u^2-1}} = \frac{1}{3}\int_1^{\sqrt{2}} \sqrt{u^2-1}\,\mathrm{d}u + \frac{1}{3}\int_1^{\sqrt{2}} \frac{\mathrm{d}u}{\sqrt{u^2-1}} = \frac{1}{3\sqrt{2}} - \frac{1}{6}\ln(1+\sqrt{2}) + \frac{1}{3}\ln(1+\sqrt{2})$$
$$= \boxed{\frac{1}{6}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)}.$$

Both the integrals above can be evaluated using the hyperbolic substitution $u = \cosh z$.

19 Problem 19

$$I_{19} = \int_1^\infty \frac{e^x + xe^x}{x^2 e^{2x} - 1}\,\mathrm{d}x.$$

We start with the substitution $u = xe^x \implies \mathrm{d}u = (e^x + xe^x)\,\mathrm{d}x$. Thus,

$$I_{19} = \int_e^\infty \frac{\mathrm{d}u}{u^2-1} = \frac{1}{2}\int_e^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right)\,\mathrm{d}u = \boxed{\frac{1}{2}\ln\left(\frac{e+1}{e-1}\right)}.$$

20 Problem 20

$$I_{20} = \int_0^\infty (80x^3 - 60x^4 + 14x^5 - x^6)e^{-x}\,\mathrm{d}x.$$

We invoke the improper integral definition of the gamma function to get

$$I_{20} = 80\Gamma(4) - 60\Gamma(5) + 14\Gamma(6) - \Gamma(7) = 80\times 3! - 60\times 4! + 14\times 5! - 6! = 480 - 1440 + 1680 - 720 = \boxed{0}.$$