SOLUTIONS TO SELECTED PUTNAM INTEGRATION PROBLEMS

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Abstract. The William Lowell Putnam Mathematical Competition, an undergraduate intercollegiate math competition often abbreviated to "The Putnam", has had some beautiful integrals show up in it. This (hopefully dynamic) article is an attempt to curate integration problems of the Putnam exam in one place.

Problem 1: 1987 – B1

Problem 1 is Problem B1 from the 1987 Putnam, which asks us to find the value of

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\, dx.$$

We use here the fact that

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx.$$

So, carrying out the change of variables $x \to 6 - x$, we have

$$I = \int_2^4 \frac{\sqrt{\ln(9-(6-x))}}{\sqrt{\ln(9-(6-x))} + \sqrt{\ln(6-x+3)}}\, dx$$

and, so,

$$I = \int_2^4 \frac{\sqrt{\ln(x+3)}}{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}}\, dx.$$

Then, we can write (since we have 2 representations for the same integral I)

$$2I = \int_2^4 \frac{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}}{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}}\, dx = \int_2^4 dx = 2.$$

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Thus,

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\, dx = \boxed{1}.$$

Bonus: Use a similar technique to evaluate

$$J = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}},$$

which was problem A3 on the 1980 Putnam.

Problem 2: 1985 – B5

Problem B5 of the 1985 Putnam asks us to find the value of

$$I = \int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\, dt.$$

The problem also states that we can assume that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

We first make the substitution $\sqrt{t} = u$ or, $\frac{dt}{2\sqrt{t}} = du$. Our integral then becomes

$$\begin{aligned} I &= 2\int_0^\infty e^{-1985(u^2+u^{-2})}\, du \\ &= 2\int_0^\infty e^{-1985(u^2+u^{-2}-2+2)}\, du \\ &= 2e^{-3970}\int_0^\infty e^{-1985(u-u^{-1})^2}\, du. \end{aligned}$$

Now, let $u^{-1} = y$ so that $-\frac{du}{u^2} = dy$. Thus,

$$I = 2e^{-3970}\int_\infty^0 e^{-1985(y^{-1}-y)^2}\, \frac{-dy}{y^2}.$$

This y is just a dummy variable, and so we can revert to u so that

$$I = 2e^{-3970}\int_0^\infty \frac{1}{u^2} e^{-1985(u-u^{-1})^2}\, du.$$

We now have two different representations for the same integral I. Adding them both, we have

$$2I = 2e^{-3970}\int_0^\infty \left(1 + \frac{1}{u^2}\right) e^{-1985(u-u^{-1})^2}\, du.$$

This is actually wonderful because we can spot a recognizable differential element in the parentheses! We can now write $\sqrt{1985}(u - u^{-1}) = x$, so that

$$\left(1 + \frac{1}{u^2}\right) du = \frac{dx}{\sqrt{1985}}.$$

Thus,

$$I = \frac{e^{-3970}}{\sqrt{1985}} \underbrace{\int_{-\infty}^{\infty} e^{-x^2}\, dx}_{\sqrt{\pi}} = \boxed{e^{-3970}\sqrt{\frac{\pi}{1985}}}.$$

Remarks: When choosing the limits of integration for the final step, we have used the fact that $\lim_{x \to 0^+} \left(x - \frac{1}{x}\right) = -\infty$ as the corresponding two-sided limit does not exist.

Additionally, this integral is also related to the modified Bessel function of the second kind, which has the integral representation

$$K_\alpha(x) = \int_0^\infty e^{-x \cosh t} \cosh(\alpha t)\, dt.$$

To see how, we start with

$$K_{1/2}(x) = \int_0^\infty e^{-x \cosh t} \cosh\left(\frac{t}{2}\right) dt,$$

or,

$$K_{1/2}(x) = \underbrace{\frac{1}{2}\int_0^\infty e^{-x(e^t + e^{-t})/2} e^{t/2}\, dt}_{I_1} + \underbrace{\frac{1}{2}\int_0^\infty e^{-x(e^t+e^{-t})/2} e^{-t/2}\, dt}_{I_2},$$

where we have used the definition of the hyperbolic cosine. Now, for I_1, we use (with slight abuse of notation) the substitution $u = e^t$ to get

$$I_1 = \frac{1}{2}\int_1^\infty e^{-x(u+u^{-1})/2} u^{1/2} \frac{du}{u} = \frac{1}{2}\int_1^\infty u^{-1/2} e^{-x(u+u^{-1})/2}\, du,$$

and the substitution $u = e^{-t}$ for the integral I_2 to get

$$I_2 = \frac{1}{2}\int_1^0 e^{-x(u+u^{-1})/2} u^{1/2} \left(\frac{-du}{u}\right) = \frac{1}{2}\int_0^1 u^{-1/2} e^{-x(u+u^{-1})/2}\, du.$$

Both I_1 and I_2 have the same integrand but the limits of integration are 0 to 1 and 1 to ∞. Therefore,

$$K_{1/2}(x) = I_1 + I_2 = \frac{1}{2}\int_0^\infty u^{-1/2} e^{-x(u+u^{-1})/2}\, du$$

and

$$2K_{1/2}(3970) = \int_0^\infty u^{-1/2} e^{-1985(u+u^{-1})}\, du = I.$$

Special functions do not often have nice closed-form expressions but we are in luck here! The modified Bessel function of the second kind admits a closed-form expression for half-odd integer orders. In particular,

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}.$$

So,

$$I = 2K_{1/2}(3970) = 2\sqrt{\frac{\pi}{2 \cdot 3970}} e^{-3970} = \boxed{e^{-3970}\sqrt{\frac{\pi}{1985}}},$$

yielding the same value as above.

Problem 3: 1997 – A3

The integral of this section is Problem A3 from the 1997 Putnam. In particular, we are asked to evaluate

$$I = \int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx.$$

This ominous-looking integral is not too hard once we realize how its different constituents fit. The infinite series in the left parenthesis is just

$$\begin{aligned} xe^{-x^2/2} &= x\left(1 - \frac{x^2}{2} + \frac{(-x^2/2)^2}{2!} + \frac{(-x^2/2)^3}{3!} + \cdots \right) \\ &= \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right). \end{aligned}$$

Therefore,

$$I = \int_0^\infty xe^{-x^2/2} \sum_{n=0}^\infty \frac{x^{2n}}{2^{2n}(n!)^2}\, dx = \sum_{n=0}^\infty \int_0^\infty xe^{-x^2/2} \frac{x^{2n}}{2^{2n}(n!)^2}\, dx,$$

where we can justify the interchange of the sum and the integral by the Monotone Convergence Theorem. Next, let us pull out the n dependent terms outside the integral to give us

$$I = \sum_{n=0}^\infty \frac{1}{2^{2n}(n!)^2} \int_0^\infty xe^{-x^2/2} x^{2n}\, dx.$$

For the integral, let us substitute $t = x^2/2$ and so $dt = x\,dx$. Thus,

$$I = \sum_{n=0}^{\infty} \frac{1}{2^{2n}(n!)^2} \int_0^{\infty} e^{-t}(2t)^n \,dt = \sum_{n=0}^{\infty} \frac{1}{2^n(n!)^2} \int_0^{\infty} e^{-t}t^n \,dt$$
$$= \sum_{n=0}^{\infty} \frac{1}{2^n(n!)^2}\Gamma(n+1),$$

using the fundamental definition of the Gamma function. For every non-negative integer n, $\Gamma(n+1) = n!$ and so

$$I = \sum_{n=0}^{\infty} \frac{1}{2^n(n!)^2} n! = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} = e^{1/2} = \boxed{\sqrt{e}}.$$

Problem 4: 1992 – A2

Next, we look at Problem A2 from the 1992 Putnam. The problem goes as follows: Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1+x)^{\alpha}$. Evaluate

$$I = \int_0^1 C(-y-1)\left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992}\right) dy.$$

For starters, let us figure out what $C(\alpha)$ looks like. Indeed,

$$C(\alpha) = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-1991)}{1992!},$$

and, so,

$$C(-y-1) = \frac{(-y-1)(-y-2)(-y-3)\cdots(-y-1992)}{1992!},$$

or,

$$C(-y-1) = \frac{(y+1)(y+2)(y+3)\cdots(y+1992)}{1992!},$$

and so,

$$C(-y-1)\left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992}\right)$$
$$= \frac{d}{dy}\left(\frac{(y+1)(y+2)(y+3)\cdots(y+1992)}{1992!}\right).$$

So our integral is just

$$\int_0^1 \frac{d}{dy}\left(\frac{(y+1)(y+2)(y+3)\cdots(y+1992)}{1992!}\right)dy$$
$$= \frac{(y+1)(y+2)(y+3)\cdots(y+1992)}{1992!}\Bigg|_0^1,$$

or,

$$I = \frac{1993! - 1992!}{1992!} = \frac{1993\cdot 1992! - 1992!}{1992!} = 1992.$$

Problem 5: 2005 – A5

Our objective for this section is to try and deal with Problem A5 from the 2005 Putnam. In particular, we are asked to evaluate

$$I = \int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx.$$

We try and evaluate the above integral by parametrizing it and then differentiating under the integral sign. Accordingly, let

$$J(t) = \int_0^1 \frac{\ln(tx+1)}{x^2+1}\,dx.$$

Then,

$$J'(t) = \int_0^1 \frac{\partial}{\partial t}\frac{\ln(tx+1)}{x^2+1}\,dx = \int_0^1 \frac{x}{(x^2+1)(tx+1)}\,dx,$$

or, decomposing into partial fractions,

$$J'(t) = \int_0^1 \left(\frac{t}{(t^2+1)(x^2+1)} + \frac{x}{(t^2+1)(x^2+1)} - \frac{t}{(t^2+1)(tx+1)}\right)dx.$$

All of the terms in the parentheses have elementary integrals, and we can write

$$J'(t) = \frac{t}{t^2+1}\arctan x + \frac{1}{2(t^2+1)}\ln(x^2+1) - \frac{1}{t^2+1}\ln(tx+1)\Bigg|_0^1$$
$$= \frac{\pi t}{4(t^2+1)} + \frac{\ln 2}{2(t^2+1)} - \frac{\ln(t+1)}{t^2+1}.$$

Now, if we integrate the RHS between 0 and t (switching the variable from t to z in the last term, to avoid confusion between the variable of integration and the parameter associated with the function $J(\cdot)$), we get

$$J(t) - J(0) = \frac{\pi}{8}\ln(t^2+1)\Big|_0^t + \frac{1}{2}\ln 2 \arctan(t)\Big|_0^t - \int_0^t \frac{\ln(z+1)}{z^2+1}\,dz.$$

Note that $J(1) = I$ and $J(0) = 0$. Therefore with $t = 1$ in the above we have,

$$I = \frac{\pi}{8}\ln(t^2+1)\Big|_0^1 + \frac{1}{2}\ln 2 \arctan(t)\Big|_0^1 - \underbrace{\int_0^1 \frac{\ln(z+1)}{z^2+1}\,dz}_{I}.$$

Hence,

$$2I = \frac{\pi}{8}\ln 2 + \frac{\pi}{8}\ln 2$$

and,

$$I = \boxed{\frac{\pi}{8}\ln 2}.$$

Remarks: This integral is called *Serret's integral*, after Joseph-Alfred Serret (of "Frenet-Serret frame" fame), who evaluated it in 1844 [1].

Bonus:

- Try evaluating the integral using the substitution $x = \tan z$.
- Try evaluating the integral using the substitution $(1+x)(1+z) = 2$.

Problem 6: 1989 – A2

Next, we try and solve Problem A2 of the 1989 Putnam. The problem asks us to find the value of

$$I = \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\,dy\,dx,$$

where $a, b > 0$.

Understanding the geometry of the region of integration is the key to solving this problem, and, indeed, we illustrate this in Figure 1. The region of integration is the rectangle $R = \{(x,y)\,|\,0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$. We divide this rectangle into two parts along the diagonal $ay = bx$. From inspection, it is clear that in the red sub-region of the rectangle, $e^{\max\{b^2x^2, a^2y^2\}} = e^{b^2x^2}$ and in the blue sub-region,

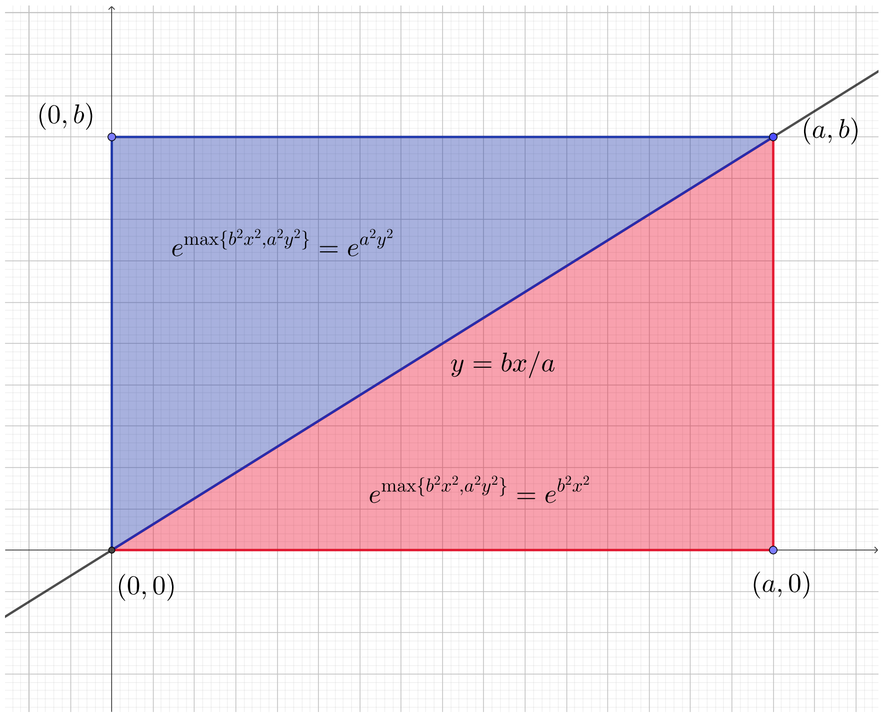

FIGURE 1. The geometry of Problem 6.

$e^{\max\{b^2x^2,a^2y^2\}} = e^{a^2y^2}$. Then, we can split I into the two double integrals

$$\begin{aligned} I &= \int_{x=0}^{x=a}\int_{y=0}^{y=bx/a} e^{b^2x^2}\,dy\,dx + \int_{y=0}^{y=b}\int_{x=0}^{x=ay/b} e^{a^2y^2}\,dx\,dy \\ &= \int_0^a \frac{bx}{a}e^{b^2x^2}\,dx + \int_0^b \frac{ay}{b}e^{a^2y^2}\,dy \\ &= \frac{1}{2ab}e^{b^2x^2}\Big|_0^a + \frac{1}{2ab}e^{a^2y^2}\Big|_0^b \\ &= \frac{e^{a^2b^2}-1}{2ab} + \frac{e^{a^2b^2}-1}{2ab} \\ &= \boxed{\frac{e^{a^2b^2}-1}{ab}}. \end{aligned}$$

Problem 7: 2016 – A3

Problem A3 from the 2016 Putnam, is a personal favorite of mine, a wonderful marriage between functional equations and integration.

Problem: Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$). Evaluate

$$\int_0^1 f(x)\, dx.$$

At the heart of this problem is actually the function

$$g(x) = 1 - \frac{1}{x}.$$

Let us iterate this map to find

$$g(g(x)) = 1 - \frac{1}{1 - \frac{1}{x}} = 1 - \frac{x}{x-1} = \frac{1}{1-x},$$

and

$$g(g(g(x))) = \frac{1}{1 - (1 - 1/x)} = x,$$

and, hence, this map is actually a 3-cycle. Sweet, because we can now write

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x, \tag{1}$$

and then first swap x with $1 - 1/x$ to write

$$f\left(1 - \frac{1}{x}\right) + f\left(\frac{1}{1-x}\right) = \arctan\left(1 - \frac{1}{x}\right), \tag{2}$$

and then once again swap x with $1 - 1/x$ to write

$$f\left(\frac{1}{1-x}\right) + f(x) = \arctan\left(\frac{1}{1-x}\right). \tag{3}$$

We now add equations (1) and (3) and then subtract equation (2) from the result to yield

$$2f(x) = \arctan x + \arctan\left(\frac{1}{1-x}\right) - \arctan\left(\frac{x-1}{x}\right). \tag{4}$$

Therefore,

$$2f(1-x) = \arctan(1-x) + \arctan\left(\frac{1}{x}\right) - \arctan\left(\frac{x}{x-1}\right). \tag{5}$$

Now, we can add equations (4) and (5) to give us

$$\begin{aligned} 2f(x) + 2f(1-x) = &\left[\arctan x + \arctan\left(\frac{1}{x}\right)\right] \\ &+ \left[\arctan\left(\frac{1}{1-x}\right) + \arctan(1-x)\right] \\ &- \left[\arctan\left(\frac{x-1}{x}\right) + \arctan\left(\frac{x}{x-1}\right)\right]. \end{aligned}$$

We note that our domain of integration is $(0,1)$, which means that the value of the expression in the last square bracket is actually $-\pi/2$ since $\arctan(x) + \arctan(1/x) = \pi/2$ if $x > 0$ and $\arctan(x) + \arctan(1/x) = -\pi/2$ if $x < 0$. Therefore,

$$f(x) + f(1-x) = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{3\pi}{4}.$$

We need to find

$$I = \int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx,$$

where the second equality is a consequence of

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

for a sufficiently well-behaved f. So then,

$$2I = \int_0^1 (f(x) + f(1-x))\,dx = \frac{3\pi}{4},$$

and, so,

$$I = \boxed{\frac{3\pi}{8}}.$$

Remarks:

(a) While utilizing the relationship between the sum of the inverse tangent of a quantity and the inverse tangent of its reciprocal is by far the most elegant way of solving this problem, one also has a gnarly way of solving this problem by directly integrating both sides of equation (4) with respect to x between 0 and 1. Indeed, if we integrate by parts, we get

$$\int_0^1 \arctan(x)\,dx = \frac{\pi}{4} - \frac{\ln(2)}{2},$$

$$\int_0^1 \arctan\left(\frac{1}{1-x}\right)\,dx = \frac{\pi}{4} + \frac{\ln(2)}{2},$$

and

$$\int_0^1 \arctan\left(\frac{x-1}{x}\right)\,dx = -\frac{\pi}{4}.$$

We then finish things off by adding the first two of the above results, subtracting the third, and (noting that we actually have obtained twice our required integral) dividing the result by 2 to yield $3\pi/8$.

(b) The map

$$g : x \mapsto \frac{x-1}{x} \coloneqq \frac{ax+b}{cx+d}$$

when $a = 1, b = -1, c = 1, d = 0$. This is what is called a linear fractional transformation. If we put the elements a, b, c, d into a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

iterating the map is equivalent to finding the powers of M. Indeed,

$$M^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

which is equivalent to the statement

$$g(g(x)) = \frac{-1}{x-1} = \frac{1}{1-x},$$

as we obtained earlier! How do we find $g^{-1}(x)$? You guessed it right, by inverting M! (Of course, the inverse needs to exist, and to guarantee that we need $\det(M) = ad - bc \neq 0$) Thus,

$$M^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

which corresponds to

$$g^{-1}(x) = \frac{1}{1-x},$$

a fact that we can corroborate by an independent calculation. Linear fractional transformations and the beautiful theory of the matrix algebra associated with them show up in all sorts of wonderful places. But that is a story for some other time!

(c) **Bonus 1:** Can you use a similar technique (iterating a map to reveal a cycle) to solve the functional equation

$$f(x) + f\left(\frac{1}{1-x}\right) = x.$$

The answer is

$$f(x) = \frac{x^3 - x + 1}{2x(x-1)}.$$

(d) **Bonus 2 (Putnam 1971 – B2):** Can you use a similar technique (iterating a map to reveal a cycle) to solve the functional equation

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x.$$

The answer is

$$f(x) = \frac{x^3 - x^2 - 1}{2x(x-1)}.$$

Problem 8: 1990 – B1

Problem B1 of the 1990 Putnam asks us to find all real-valued continuously differentiable functions f on the real line such that for all x

$$(f(x))^2 = \int_0^x \left((f(t))^2 + (f'(t))^2\right)\, dt + 1990. \qquad (\dagger)$$

We start with differentiating both sides of the above condition. This gives us

$$2f(x)f'(x) = (f(x))^2 + (f'(x))^2,$$

which means that

$$(f(x))^2 + (f'(x))^2 - 2f(x)f'(x) = 0 \implies (f(x) - f'(x))^2 = 0.$$

This immediately implies that

$$f'(x) = f(x),$$

and the only solutions of the above differential equation are

$$f(x) = Ce^x.$$

What remains is to evaluate the constant C. From ($\dagger$), we have that $(f(0))^2 = 1990$, or, $f(0) = \pm\sqrt{1990}$. But $f(0) = C$. Hence, the only real-valued continuously differentiable functions f on the real line that satisfy ($\dagger$) are

$$f(x) = \pm\sqrt{1990}e^x.$$

Problem 9: 1993 – A5

Again, this is probably one of my most favorite problems from the Putnam. Problem A5 from 1993 asks us to show that

$$\int_{-100}^{-10}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx+\int_{\frac{1}{101}}^{\frac{1}{11}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx+\int_{\frac{101}{100}}^{\frac{11}{10}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx$$

is a rational number.

As in Problem 7, the trick here is find a transformation for which the integrand is invariant under the transformation. If

$$g(x)=\left(\frac{x^2-x}{x^3-3x+1}\right)^2,$$

then,

$$g\left(\frac{1}{1-x}\right)=g\left(1-\frac{1}{x}\right)=g(x).$$

Define

$$I=\int_{-100}^{-10}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx,$$

$$J=\int_{\frac{1}{101}}^{\frac{1}{11}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx,$$

and

$$K=\int_{\frac{101}{100}}^{\frac{11}{10}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx.$$

Carrying out the substitution $x\mapsto 1-1/x$ in J gives us

$$J=\int_{-100}^{-10}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 \frac{dx}{x^2},$$

and carrying out the substitution $x\mapsto 1/(1-x)$ in K gives us

$$K=\int_{-100}^{-10}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 \frac{dx}{(1-x)^2}.$$

Then,

$$\begin{aligned} I + J + K &= \int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \left(1 + \frac{1}{x^2} + \frac{1}{(1-x)^2} \right) dx \\ &= \int_{-100}^{-10} \frac{\left(1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right)}{\left(\frac{x^3-3x+1}{x^2-x}\right)^2} dx \\ &= \int_{-100}^{-10} \frac{\left(1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right)}{\left(1 + x - \frac{1}{x} - \frac{1}{x-1}\right)^2} dx \\ &= -\frac{x^2 - x}{x^3 - 3x + 1} \bigg|_{-100}^{-10}, \end{aligned}$$

which is obviously a rational number. (The actual value of the sum of the three integrals is $\frac{11131110}{107634259}$).

Problem 10: 1984 – A5

Problem: Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \leqslant 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral

$$\iiint_R x^1 y^9 z^8 w^4 \, dx \, dy \, dz$$

in the form $a!b!c!d!/n!$, where a, b, c, d, and n are positive integers.

A neat approach from the official solutions is to generalize this problem. Indeed, let R_t be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \leqslant t$. Define

$$I(t) = \iiint_{R_t} x^1 y^9 z^8 (t - x - y - z)^4 \, dx \, dy \, dz.$$

We perform a scaling of variables $x = tu, y = tv$, and $z = tw$. This yields

$$\begin{aligned}
I(t) &= \iiint_{R_t} (tu)(tv)^9(tw)^8(t - tu - tv - tw)^4 \, tdu \, tdv \, tdw \\
&= \iiint_{R_t} (tu)(tv)^9(tw)^8 t^4 (1 - u - v - w)^4 t^3 \, du \, dv \, dw \\
&= t^{25} \iiint_{R_t} u^1 v^9 w^8 (1 - u - v - w)^4 \, du \, dv \, dw \\
&= t^{25} I(1).
\end{aligned}$$

Define, then

$$J = \int_0^\infty I(t)e^{-t} \, dt = \int_0^\infty t^{25} I(1) e^{-t} \, dt = I(1) \int_0^\infty t^{25} e^{-t} \, dt = I(1)\Gamma(26) = I(1) \cdot 25!.$$

However, it is also true that

$$J = \int_0^\infty \iiint_{R_t} e^{-t} x^1 y^9 z^8 (t - x - y - z)^4 \, dx \, dy \, dz \, dt.$$

Perform the substitution $s = t - x - y - z$. This means that $t = s + x + y + z$. Therefore,

$$\begin{aligned}
J &= \int_0^\infty \iiint_{R_t} e^{-t} x^1 y^9 z^8 (t - x - y - z)^4 \, dx \, dy \, dz \, dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-s} e^{-x} e^{-y} e^{-z} x^1 y^9 z^8 s^4 \, dx \, dy \, dz \, ds \\
&= \left(\int_0^\infty x^1 e^{-x} \, dx \right) \left(\int_0^\infty y^9 e^{-y} \, dy \right) \left(\int_0^\infty z^8 e^{-z} \, dz \right) \left(\int_0^\infty s^4 e^{-s} \, ds \right) \\
&= \Gamma(2)\Gamma(10)\Gamma(9)\Gamma(5) \\
&= 1!9!8!4!.
\end{aligned}$$

Our problem actually needs us to find $I(1)$. But

$$I(1) = \frac{J}{25!} = \boxed{\frac{1!9!8!4!}{25!}}.$$

Problem 11: 1982 – A3

We are asked to evaluate the integral

$$I = \int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx.$$

We try Feynman's favorite method, parametrizing the integral and differentiating under the integral sign. Accordingly, let

$$J(t) = \int_0^\infty \frac{\arctan(tx) - \arctan(x)}{x}\, dx.$$

Then,

$$\begin{aligned} J'(t) &= \int_0^\infty \frac{\partial}{\partial t} \frac{\arctan(tx) - \arctan(x)}{x}\, dx \\ &= \int_0^\infty \frac{dx}{1 + t^2x^2} \\ &= \frac{1}{t^2} \int_0^\infty \frac{dx}{x^2 + \frac{1}{t^2}} \\ &= \frac{\arctan(tx)}{t}\Big|_0^\infty \\ &= \frac{\pi}{2t}. \end{aligned}$$

Now, $J(1) = 0$. Therefore, integrating both sides between the limits 1 and t yields

$$J(t) = \frac{\pi}{2} \int_1^t \frac{dz}{z} = \frac{\pi}{2} \ln(t).$$

We then finish off by recognizing that $I = J(\pi)$, and writing

$$I = J(\pi) = \boxed{\frac{\pi}{2} \ln(\pi)}.$$

References

[1] Serret, Joseph-Alfred. "Sur l'intégrale $\int_0^1 \frac{\ell(1+x)}{1+x^2}\, dx$." *J. Math. Pures Appl.* 1.9 (1844): 436.