A Separation Principle for Discrete-Time Fractional-Order Dynamical Systems and Its Implications to Closed-Loop Neurotechnology

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***Abstract*—Closed-loop neurotechnology requires the capability to predict the state evolution and its regulation under (possibly) partial measurements. There is evidence that neurophysiological dynamics can be modeled by fractional-order dynamical systems (FODS). Therefore, we propose to establish a separation principle for discrete-time FODS, which are inherently nonlinear and are able to capture spatiotemporal relations that exhibit non-Markovian properties. The separation principle states that the problems of controller and state estimator design can be done independently of each other while ensuring proper estimation and control in closed-loop setups. Lastly, we illustrate, as proof-of-concept, the application of the separation principle when designing controllers and estimators for these classes of systems in the context of neurophysiological data. In particular, we rely on real data to derive the models used to assess and regulate the evolution of closed-loop neurotechnologies based on electroencephalographic data.**

***Index Terms*—Control applications, biomedical, estimation.**

I. Introduction

THERE is an increasing trend of looking into leveraging *closed-loop* control and estimation strategies for the continued monitoring and interaction of subjects in the form of closed-loop neurotechnologies. Such technologies bring the promise of improving the quality-of-life of patients affected by neurological disorders such as epilepsy [1], Parkinson's disease [2], Alzheimer's disease [3], anxiety [4], and depression [5]. For neurophysiological signals, lingering interacting effects originating from long-term temporal dependence properties have illustrated the potential for clinical applications of *fractional-order* based modeling, design, and analysis of such neurotechnologies [6]–[9].

Due to the highly dynamic nature of the neurophysiological processes, it is imperative that we consider *feedback* mechanisms [10]. A particularly successful closed-loop controller design strategy that has achieved remarkable success in several engineering applications is the strategy of *model predictive control* (MPC) [11]. Indeed, the main advent of model-based approaches is that we can understand how an external signal or stimulus would craft the dynamics of the process. In [12], the authors propose an electrical neurostimulation MPC-based strategy for the mitigation of epileptic seizures by modeling brain dynamics through fractional-order systems.

Recent work provides evidence that fractional-order dynamical systems (FODS) exhibit great success in accurately modeling dynamics which undergo nonexponential power-law decay, and have long-term memory or fractal properties [13]–[18]. Not only have FODS found applications in domains such as gas dynamics [19], viscoelasticity [20], chaotic systems [21], and biological swarming [22], just to mention a few, but also in cyber-physical systems to model the interlaced evolution of the spatial and temporal components of complex networks [23], [24]. Some of these relationships have also been explored in the domain of neurophysiological signals such as electroencephalogram (EEG) and electrocardiogram (ECG) [25].

The *separation principle*, one of the cornerstones of modern feedback systems theory, states that the problems of optimal control and state estimation can be decoupled in certain specific instances [26]. These ideas were advanced early on in [27], [28], and [29] and are connected to the idea of *certainty equivalence* [30] in stochastic control theory. Since then, the separation principle has been proposed in a wide variety of settings, including, but not limited to, stochastic control systems [31], [32], hybrid systems [33], distributed control systems [34], quantum control [35], linear systems with Markovian jumps [36], wireless fading channels subject to channel capacity constraints [37], and discrete-time networked control systems with random packet drops [38].

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Although separation principle-like results have been proposed for nonlinear systems, e.g., in [39] and [40], the separation principle does not hold for nonlinear systems in general. Therefore, in this letter, we *state and prove a separation principle result that stems from the problem of closed-loop discrete-time FODS and demonstrate the implications of our results in the context of closed-loop neurotechnology using real-world electroencephalographic data*. Specifically, we prove that if a closed-loop controller and an observer are designed for discrete-time FODS, then the aforementioned design can be carried out independently of each other. FODS are inherently nonlinear and they possess long-term memory in the sense that the evolution of a FODS aggregates the effects of *all time* as the evolution of the system progresses. As a consequence, the innate non-Markovian nonlinearity of FODS does not immediately ensure the existence of a separation simple for the reasons mentioned above. Furthermore, FODS are finding increasing applications in the field of MPC, where the problems of estimator and controller design need the existence of a separation principle. Although separation principle results such as [41]–[43] have been derived for FODS in continuous time, to the best of our knowledge, no such result has been previously proposed and analyzed for discrete-time FODS.

The remainder of this letter is organized as follows. Section II presents some essential theory regarding discrete-time FODS including the system model that we consider and the separation principle we propose to prove. Section III presents the proof of the separation principle for discrete-time FODS. Section IV provides an illustrative example that shows how the separation principle can be used to sustain closed-loop feedback performance in the context of neurotechnology, and Section V concludes this letter.

II. Problem Statement

We consider a deterministic linear discrete-time fractional-order dynamical system described as follows

$$\begin{aligned}\Delta^{\alpha} x[k+1] &= Ax[k] + Bu[k]\\ y[k] &= Cx[k]\\ x[0] &= x_0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state* for time step $k \in \mathbb{N}$, $u \in \mathbb{R}^p$ is the *input* and $y \in \mathbb{R}^n$ is the *output*. $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times p}$ is the input matrix, and $C \in \mathbb{R}^{p\times n}$ is the sensor measurement matrix. Note that the system model is similar to a classic discrete-time linear time-invariant model but it is nonlinear due to the inclusion of the fractional derivative, whose expansion and discretization for the i-th state, $1 \leq i \leq n$, can be written as

$$\Delta^{\alpha_i} x_i[k] = \sum_{j=0}^{k} \psi(\alpha_i, j) x_i[k-j], \tag{2}$$

where α_i is the fractional order corresponding to state i and

$$\psi(\alpha_i, j) = \frac{\Gamma(j-\alpha_i)}{\Gamma(-\alpha_i)\Gamma(j+1)}, \tag{3}$$

with $\Gamma(\cdot)$ being the gamma function defined by $\Gamma(z) = \int_0^\infty s^{z-1} e^{-s}\, \mathrm{d}s$ for all complex numbers z with $\Re(z) > 0$ [10].

Given the deterministic linear discrete-time fractional-order dynamical system (1), we have two main control objectives that need to be satisfied.

- *Stabilizability:* In this problem, we deal with the issue of *stabilization* of system (1). To this end, we consider the problem of designing a controller to stabilize the system (1). The second control objective is concerned with designing an observer for (1).
- *Observer Design:* Assume that the states of (1) are not known exactly. In this problem, we deal with the issue of designing an *observer* for the system (1). The observer that we design should help us to estimate the states of the system given knowledge of the input $u \in \mathbb{R}^p$ and the output $y \in \mathbb{R}^n$.

With these two objectives in mind, we seek to prove the following result.

Problem 1: Given the deterministic linear discrete-time fractional-order dynamical system (1), can the problems of stabilizability and observer design can be carried out independently of each other towards achieving closed-loop stabilizability with partial measurements?

III. Separation Principle for Fractional-Order Systems

In this section, we will present the main result of this letter, i.e., the separation principle for discrete-time FODS. We first introduce the theory of state evolution in discrete-time FODS, presenting the relevant equations for the evolution of the dynamics of the system states in Lemma 1. We will then sequentially consider the problems of observer design (in Section III-A), which entails the construction of an observer for the dynamical system (1), followed by the problem of stabilizability (in Section III-B), which requires us to design a controller to stabilize the system (1). With the above ingredients, and some mathematical preliminaries, we present the statement and proof of the main result of this letter, the separation principle for discrete-time FODS in Section III-C (see Theorem 1).

We begin by reviewing some essential theory for fractional-order systems, including closed-form expressions for the state dynamics. Using the expansion of the fractional-order derivative in (2), the evolution of the state vector can be written as follows

$$\begin{aligned} x[k+1] &= Ax[k] - \sum_{j=1}^{k+1} D(\alpha, j) x[k+1-j] + Bu[k]\\ x[0] &= x_0, \end{aligned} \tag{4}$$

where $D(\alpha, j) = \operatorname{diag}(\psi(\alpha_1, j), \psi(\alpha_2, j), \ldots, \psi(\alpha_n, j))$. Alternatively, (4) can be written as

$$\begin{aligned} x[k+1] &= \sum_{j=0}^{k} A_j x[k-j] + Bu[k]\\ x[0] &= x_0, \end{aligned} \tag{5}$$

where $A_0 = A - D(\alpha, 1)$ and $A_j = -D(\alpha, j+1)$ for $j \geq 1$. Defining matrices G_k as

$$G_k = \begin{cases} I_n & k = 0, \\ \sum_{j=0}^{k-1} A_j G_{k-1-j} & k \geq 1, \end{cases} \tag{6}$$

we can state the following result.

Lemma 1 [44]: The solution to the system described by (1) is given by

$$x[k] = G_k x[0] + \sum_{j=0}^{k-1} G_{k-1-j} B u[j]. \tag{7}$$

Having obtained the closed-form expressions for the state vectors, we turn our attention to the problems of observer design and stabilizability.

A. Observer Design

In this section, we will show that it is possible to obtain an unbiased estimator by considering an innovation term added to the dynamics of the state estimate, which can be described as follows. We consider the construction of a Luenberger-like observer [45], whose state and output estimates are denoted by $\hat{x}[k] \in \mathbb{R}^n$ and $\hat{y}[k] \in \mathbb{R}^n$, respectively. This observer then takes the following form

$$\begin{aligned} \hat{x}[k+1] &= \sum_{j=0}^{k} A_j \hat{x}[k-j] + Bu[k] + L(y[k] - \hat{y}[k]), \\ \hat{y}[k] &= C\hat{x}[k], \end{aligned} \tag{8}$$

where the matrix $L \in \mathbb{R}^{n \times n}$ is a weighting matrix that weights the difference between the outputs of the plant and the observer. Implicitly, we assume that (A, C) is detectable, such that we can design a matrix L so that the closed-loop observer poles lie within the unit circle. Note that the observer consists of two parts, the first part being a copy of the plant's dynamics as applied to the observer, and an innovation term being a scaled version of the difference between the outputs of the plant and the observer.

B. Stabilizability and Output Feedback

In this section, we consider the problem of stabilizing (5) in a classical state-feedback control setting by adopting the following steps:

(i) Deriving an expression for the control input in terms of the states of the observer with memory.

(ii) Using the above expression for the control input to address the problem of output feedback and, in particular, deriving the dynamics of the estimation error signal.

To derive an expression for the control input in terms of the states of the observer with memory, let us assume that the control input $u \in \mathbb{R}^p$ can be written a weighted linear combination of the states of the observer with memory, i.e.,

$$\begin{aligned} u[k] &= F_0 \hat{x}[k] + F_1 \hat{x}[k-1] + \ldots + F_k \hat{x}[0] \\ &= \sum_{j=0}^{k} F_j \hat{x}[k-j], \end{aligned} \tag{9}$$

where $F_j \in \mathbb{R}^{p \times n}$ for $j = 0, 1, \ldots, k$. Substituting this into (8) and using the fact that $y[k] = Cx[k]$ and $\hat{y}[k] = C\hat{x}[k]$, we have

$$\begin{aligned} \hat{x}[k+1] &= \sum_{j=0}^{k} (A_j + BF_j)\hat{x}[k-j] + L(y[k] - \hat{y}[k]) \\ &= \sum_{j=0}^{k} (A_j + BF_j)\hat{x}[k-j] + L(Cx[k] - C\hat{x}[k]) \\ &= \sum_{j=0}^{k} (A_j + BF_j)\hat{x}[k-j] + LCe[k], \end{aligned} \tag{10}$$

where $e[k] = x[k] - \hat{x}[k]$ is defined as the error between the states of the plant and the observer.

Next, we use the above expression for the control input to address the problem of output feedback and, in particular, to derive the dynamics of the estimation error signal. Going back to the dynamics of the plant and substituting (9) into (5), we have

$$\begin{aligned} x[k+1] &= \sum_{j=0}^{k} A_j x[k-j] + Bu[k] \\ &= \sum_{j=0}^{k} A_j x[k-j] + \sum_{j=0}^{k} BF_j \hat{x}[k-j] \\ &= \sum_{j=0}^{k} A_j x[k-j] + \sum_{j=0}^{k} BF_j (x[k-j] - e[k-j]) \\ &= \sum_{j=0}^{k} (A_j + BF_j) x[k-j] - \sum_{j=0}^{k} BF_j e[k-j]. \end{aligned} \tag{11}$$

Next, we consider the dynamics of the error signal. Indeed, we have

$$\begin{aligned} e[k+1] &= x[k+1] - \hat{x}[k+1] \\ &= \sum_{j=0}^{k} (A_j + BF_j) x[k-j] - \sum_{j=0}^{k} BF_j e[k-j] \\ &\quad - \left(\sum_{j=0}^{k} (A_j + BF_j)\hat{x}[k-j] + LCe[k] \right) \\ &= \sum_{j=0}^{k} (A_j + BF_j) e[k-j] - \sum_{j=0}^{k} BF_j e[k-j] \\ &\quad - LCe[k] \\ &= \sum_{j=0}^{k} A_j e[k-j] - LCe[k]. \end{aligned} \tag{12}$$

C. Separation Principle for Discrete-Time FODS

Having derived the expressions for the dynamics of the plant state and the error signal, we are now ready to state and prove the separation principle for discrete-time FODS. We first state some mathematical preliminaries that will aid our proof.

Definition 1: A Hilbert space is a vector space $\mathscr{H}$ over $\mathbb{R}$ or $\mathbb{C}$ together with an inner product $\langle \cdot, \cdot \rangle$, such that relative to the metric $d(x, y) = \|x-y\|$ induced by the norm $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$, $\mathscr{H}$ is a complete metric space.

Definition 2: The sequence space $\ell^2(\mathbb{N})$ denotes the Hilbert space of all square-summable sequences. Such sequences are represented by vectors with infinitely many elements $\mathcal{X} = \{x[0], x[1], x[2], \ldots\}$. For $\mathcal{X}, \mathcal{Y} \in \ell^2(\mathbb{N})$, the space is equipped with the inner product

$$\langle \mathcal{X}, \mathcal{Y}\rangle = \sum_{k=0}^{\infty} x[k]y[k]^*,$$

where the * denotes the complex conjugate. In other words, a sequence $\mathcal{X} \in \ell^2(\mathbb{N})$ if $\|\mathcal{X}\|^2 = \langle\mathcal{X},\mathcal{X}\rangle = \sum_{k=0}^{\infty} |x[k]|^2 < \infty$.

Definition 3: For a causal sequence $\mathcal{X}$, we define the *backward shift operator* $\mathcal{S} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$\mathcal{S}\mathcal{X} = \mathcal{S}\{x[0], x[1], x[2], \ldots\} = \{x[1], x[2], x[3], \ldots\}.$$

Definition 4: The *point spectrum* of an operator T, denoted by $\mathsf{spec}(T)$ is the set of eigenvalues of the operator T.

Lastly, we present the main result of this letter.

Theorem 1 (Separation Principle for Discrete-Time FODS): Consider the discrete-time fractional-order dynamical system given in (1), and consider the problems of

1) Designing an unbiased estimator (of the form (8)) for the system (1) by following the procedure outlined in Section III-A, and,
2) Designing a controller (of the form (9)) that stabilizes the system (1) by following the procedure outlined in Section III-B.

Then, given knowledge of the input $u \in \mathbb{R}^p$ and the output $y \in \mathbb{R}^n$, the above designs can be done independently of each other towards achieving closed-loop stabilizability with partial measurements.

Proof: With respect to our problem, we define the infinite column sequences

$$\mathscr{X} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \end{bmatrix}, \quad \mathscr{E} = \begin{bmatrix} e[0] \\ e[1] \\ e[2] \\ \vdots \end{bmatrix}. \tag{13}$$

Using $\mathscr{X}$ and $\mathscr{E}$, we can now compactly write equations (11) and (12) as follows

$$\begin{bmatrix} \mathcal{S}\mathscr{X} \\ \mathcal{S}\mathscr{E} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathscr{J}_1 & \mathscr{J}_2 \\ \mathbf{0} & \mathscr{J}_3 \end{bmatrix}}_{\mathscr{J}} \begin{bmatrix} \mathscr{X} \\ \mathscr{E} \end{bmatrix}, \tag{14}$$

where $\mathcal{S}$ is the backward shift operator and the matrices $\mathscr{J}_i$ $(i = 1, 2, 3)$ are Toeplitz with the following structures

$$\mathscr{J}_1 = \begin{bmatrix} A_0 + BF_0 & 0 & 0 & \ldots \\ A_1 + BF_1 & A_0 + BF_0 & 0 & \ldots \\ A_2 + BF_2 & A_1 + BF_1 & A_0 + BF_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{15a}$$

$$\mathscr{J}_2 = \begin{bmatrix} -BF_0 & 0 & 0 & \ldots \\ -BF_1 & -BF_0 & 0 & \ldots \\ -BF_2 & -BF_1 & -BF_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{15b}$$

$$\mathscr{J}_3 = \begin{bmatrix} A_0 - LC & 0 & 0 & \ldots \\ A_1 - LC & A_0 - LC & 0 & \ldots \\ A_2 - LC & A_1 - LC & A_0 - LC & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{15c}$$

Note that $\mathscr{J}_1$ only contains terms that pertain to the stabilizability, and $\mathscr{J}_3$ only contains terms that pertain to the observer design. From the block structure of (14), it can be seen that

$$\mathsf{spec}(\mathscr{J}) = \mathsf{spec}(\mathscr{J}_1) \cup \mathsf{spec}(\mathscr{J}_3),$$

and the design of $\mathscr{J}_1$ and $\mathscr{J}_3$ can be carried out independently of each other. ∎

Although the design of the Luenberger-like observer in Section III-A starts with the design of a single weighting matrix L that weights the outputs of the plant and the observer without memory, it is instructive to note that the separation principle for discrete-time FODS that we proved in Section III-C also holds for an observer of the following form

$$\begin{aligned} \hat{x}[k+1] &= \sum_{j=0}^{k} A_j \hat{x}[k-j] + Bu[k] \\ &\quad + \sum_{j=0}^{k} L_j(y[k-j] - \hat{y}[k-j]), \\ \hat{y}[k] &= C\hat{x}[k], \end{aligned} \tag{16}$$

with the observer weighting matrices L_j, $j = 0, \ldots, k$ being designed so as to place the closed-loop observer poles within the unit circle. The key difference between the observers in equations (8) and (16) are that in the former we have a single weighting matrix that weights the difference of the outputs of the plant and the observer, and in the latter, we use multiple weighting matrices to weight the differences of the outputs of the plant and the observer with memory. We then have the following theorem.

Theorem 2: Consider the discrete-time fractional-order dynamical system given in (1), and consider the problems of

1) Designing an unbiased estimator (of the form (16)) for the system (1) by following the procedure outlined above, and,
2) Designing a controller (of the form (9)) that stabilizes the system (1) by following the procedure outlined in Section III-B.

Then, given knowledge of the input $u \in \mathbb{R}^p$ and the output $y \in \mathbb{R}^n$, the above designs can be done independently of each other towards achieving closed-loop stabilizability with partial measurements.

Proof: By setting $L_0 = L$, and $L_1 = L_2 = \ldots = L_k = 0$ for $k = 1, 2, \ldots$, the problem reduces to the statement of Theorem 1, and the proof follows by a similar line of reasoning. ∎

IV. Closed-Loop Neurotechnology

In this section, we illustrate our results by designing a *model predictive controller* (MPC) that simulates a simple implantable closed-loop electrical neurostimulator. The controller is implemented on a discrete-time fractional-order plant, representing normal brain activity, whereas the predictive model will be based on an autoregressive finite-history approximation. Naturally, the controller will be designed as if it had access to the actual state of the system, and similarly the state observer (whose estimates are fed into the designed controller) is designed without consideration of the control strategy adopted.

We start by identifying the spatial and temporal parameters A and α in (1), from a 4-channel sample of length 1 second of normalized EEG recordings. We model the $n = 4$ components of the state vector as denoting the different recorded channels (i.e., readings obtained from microelectrodes). The data used for these experiments are from subject 11 from the CHB-MIT Scalp EEG database [46]. To achieve this identification, we leveraged the tools developed in [47], which led us to

$$A = \begin{bmatrix} 0.0350 & 0.0526 & -0.0034 & -0.0391 \\ 0.0296 & -0.0496 & 0.0646 & 0.0610 \\ -0.0103 & -0.0028 & -0.0091 & 0.0068 \\ -0.0291 & 0.0143 & -0.0008 & 0.0394 \end{bmatrix} \quad (17)$$

and

$$\alpha = \begin{bmatrix} 0.5945 & 0.7176 & 0.9603 & 0.6279 \end{bmatrix}^\mathsf{T}, \quad (18)$$

as the main parameters in the system. We are interested in modeling the impact of an electrical stimulation signal $u[k]$ originating from an integrated arbitrary voltage generator circuit. We start by considering the scenario $B = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$ corresponding to a stimulus that perturbs all channels uniformly (e.g., if the four electrodes are placed considerably near each other). The measurements $y[k]$ used to estimate the state (through a simple Kalman-like filter) will be assumed simply as those given directly by the first channel, i.e., $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

At each step k, the MPC controller will minimize a quadratic cost function

$$J(u[k], \ldots, u[k+P-1]) = \sum_{j=1}^{P} \|x[k] - x_{\text{ref}}[k+j]\|^2, \quad (19)$$

with the predicted evolution $x[k]$ evolving not by the original system (1), but instead by a multivariate autoregressive (MVAR) approximation

$$x[k+1] = \sum_{j=0}^{p-1} A_j x[k-j] + Bu[k], \quad (20)$$

based on (5), by clipping off the infinite-horizon memory dependence by instead only a p-horizon one. The *prediction horizon* P was set to $P = 8$ (50 milliseconds), whereas the *control horizon* M upon which the solution is implemented was set to $M = 4$ (25 milliseconds). The value of p was selected ad hoc to obtain a desirable performance. The reference signal $x_{\text{ref}}[k]$ denotes a simple rectangular pulse of frequency 8 Hz, within the usual range of *alpha rhythms* that characterize relaxed, but conscious brain activity [48].

The results can be seen in Fig. 1, and, as we can see, the controller is largely successful despite never having direct access to the state of the system. In other words, efficient

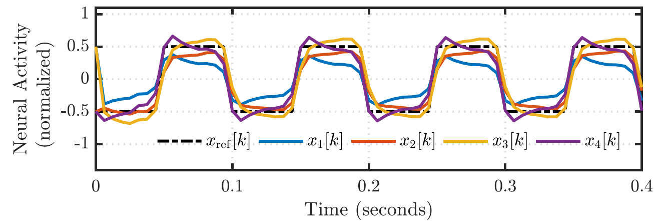

Fig. 1. MPC-based neuromodulation of a discrete-time FODS representing normal brain activity by state tracking of a rectangular pulse of frequency 8 Hz, roughly in the range of alpha rhythms.

design of a closed-loop controller and observer can be carried out separately for discrete-time fractional-order systems, as formally established in this letter.

V. Conclusion and Future Work

In this letter, we proposed and proved a separation principle result for discrete-time FODS. As a consequence, we can decouple the problems of designing, respectively, a controller for the stabilization of the system states, and an observer for the estimation of the system states. The ability of discrete-time FODS to model complex spatiotemporal relationships in neurophysiological signals have led to the use of these models in closed-loop neurotechnologies.

Very rarely in practical settings, however, do we have deterministic fractional-order models. EEG signals, for instance, are particularly prone to disturbances arising from outside the brain, which are referred to as *artifacts* in the neuroscience literature [49]. Furthermore, stabilizing these models in the presence of disturbances becomes relevant in the treatment of disorders like epilepsy, Parkinson's disease, or Alzheimer's disease, since, in recent years, there have been increasing research efforts into finding possible palliative therapies for the aforementioned using neurofeedback [50]. Future work, therefore, will focus on developing controllers and observers for FODS with associated process and measurement noise, and investigating the possible existence of separation principle-like results akin to those already existing in the field of linear stochastic control theory.

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