Dealing with State Estimation in Fractional-Order Systems under Artifacts

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Abstract— Fractional-order dynamical systems are used to describe processes that exhibit long-term memory with power-law dependence. Notable examples include complex neurophysiological signals such as electroencephalogram (EEG) and blood-oxygen-level dependent (BOLD) signals. When analyzing different neurophysiological signals and other signals with a different origin (for example, biological systems), we often find the presence of artifacts, that is, recorded activity due to external causes having origins outside the system of interest. In this paper, we consider the problem of estimating the states of a discrete-time fractional-order dynamical system when there are artifacts present in some of the sensor measurements. Specifically, we provide necessary and sufficient conditions that ensure we can retrieve the system states even in the presence of artifacts. We provide a state estimation algorithm that can estimate the states of the system in the presence of artifacts. Finally, we present illustrative examples of our main results using real EEG data.

I. INTRODUCTION

Fractional-order dynamical systems (FODS) have been successfully used to accurately model dynamics which undergo nonexponential power-law decay, and have long-term memory or fractal properties [1]–[5]. In particular, fractional-order models have been used in domains such as gas dynamics [6], viscoelasticity [7], chaotic systems [8], and biological swarming [9], just to mention a few. With the advent of cyber-physical systems (CPS), the usefulness of FODS becomes even more apparent, since we have to model the relationship between the spatial and temporal evolution of complex networks [10], [11].

Inspired by the recent spate of application of FODS to model the spatiotemporal properties of complex physiological signals such as electroencephalogram (EEG), electrocardiogram (ECG), electromyogram (EMG), and blood-oxygen-level dependent (BOLD) imaging [12], [13] in the context of neurophysiological applications, the estimation of the states of a fractional-order dynamical system plays a key role in assessing how the corresponding states are evolving, which unveil information about brain function. We are particularly interested in EEG signals, that are known to be prone to disturbances which are not cerebral in origin, which are known as *artifacts* in the neuroscience literature [14]. There are innumerable sources for these artifacts, some of the common ones being artifacts due to the blinking of the

eye (called blink artifact), the potential difference between the EEG electrode and its lead (called electrode pop artifact), and muscular motions of chewing or swallowing (called glossokinetic artifact). Therefore, it becomes imperative to develop a set of tools which can robustly estimate the dynamics of the EEG signals even in the presence of these artifacts. Note that although the term 'artifact' is almost always used to describe behavior which is not of a cerebral nature in EEG signals, the notion of artifacts can be readily extended to characterize the disturbances that are not consistent with the stationary evolution of the system's dynamics - in particular, FODS.

In this paper, we develop a systematic framework to estimate the states of a fractional-order dynamical system when artifacts are associated with the measurement process. To do this, we are inspired by recent research on resilient state estimation of dynamical systems in the presence of attacks. For instance, [15]–[26] deal with the problem of identification and state estimation of cyber-physical systems in the presence of these attacks. Notice that artifacts are mainly 'attacks by nature', which we try to cope in the context of FODS, and are not assumed to follow any stochastic process or dynamics. Hence, inspired by the techniques outlined in the former, we develop a method to estimate the states of a fractional-order dynamical system in the presence of artifacts, which, to the best of our knowledge, has not been previously analyzed in the context of FODS.

A few works already exist in the domain of state estimation of fractional-order systems [27]–[31]. However, most of these assume the noise associated with the measurement process to have a certain stochastic characterization. The work that comes closest to ours is [31], but even that outlines a method for artifact correction using fractional calculus and a median filter.

The main theoretical contributions of our paper are as follows. First, we provide necessary and sufficient conditions that ensure the estimation of the states of a discrete-time fractional-order dynamical system in the presence of arbitrary artifacts. Next, we propose an algorithm that can estimate the states in the presence of artifacts in the sensor measurements. Finally, we borrow techniques of compressive sensing to improve the algorithm such that it becomes computationally feasible.

The remainder of the paper is organized as follows. Section II introduces the elements of the state estimation problem. Section III presents necessary and sufficient conditions that guarantee the estimation of states of FODS in the presence of artifacts. Sections IV and V consider

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the problems of constructing the state estimator using the techniques of compressive sensing. Finally, in Section VI, we present illustrative examples showing the performance of our algorithm on a pedagogical example, as well as real data collected from a wearable EEG device. The detailed proofs of the results presented in this paper can be found in the extended version of the paper in [32].

II. PROBLEM STATEMENT

A. System Model

Consider a linear discrete-time fractional-order dynamical model described as follows

$$\Delta^{\alpha} x[k+1] = Ax[k]$$
$$y[k] = Cx[k] + e[k], \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state for time step $k \in \mathbb{N}$ and $y \in \mathbb{R}^p$ is the output vector. $A \in \mathbb{R}^{n \times n}$ is the system matrix and $C \in \mathbb{R}^{p \times n}$ is the sensor measurement matrix. The vector $e \in \mathbb{R}^p$ represents the model of errors, i.e., the *artifacts*. Note that for the special case of EEG signals, these could be some of the EEG artifacts described in the Introduction. If there are no artifacts in channel $i \in \{1, \ldots, p\}$, then $e_i[k] = 0$ and the output $y_i[k]$ of channel i is not corrupted. As a consequence, the sparsity pattern of $e[k]$ gives the set of sensors where there are artifacts present. Note that the system model is similar to a classic discrete-time linear time-invariant model except for the inclusion of the fractional derivative, whose expansion and discretization for the i-th state, $1 \le i \le n$, can be written as

$$\Delta^{\alpha_i} x_i[k] = \sum_{j=0}^{k} \psi(\alpha_i, j) x_i[k-j], \qquad (2)$$

where α_i is the fractional order corresponding to state i and

$$\psi(\alpha_i, j) = \frac{\Gamma(j - \alpha_i)}{\Gamma(-\alpha_i)\Gamma(j+1)}, \qquad (3)$$

with $\Gamma(\cdot)$ being the gamma function [33].

B. Artifact Model

We assume in this paper that the measurement channels in which the artifacts are present do not change over time, but can be arbitrary. Note that this is a realistic assumption for our problem since neurophysiological studies of EEG artifacts indicate that they generally occur in specific channels without switching channels arbitrarily and have highly conserved morphological structure [34]. Therefore, we will talk about the *number* of channels which are affected by artifacts, which will allow us to make statements about unambiguously recovering the initial state $x[0]$ even in the presence of a given number of artifacts.

C. Central Estimator Model

We will also assume the presence of a central estimator, whose job is to receive the output $y_i[k]$, $i \in \{1, \ldots, p\}$ of each sensor at every time step k and, from this information, estimate the initial state $x[0]$. We also assume that the estimator has knowledge of the system matrix A, the fractional-order coefficients $\{\alpha_i\}_{i=1}^n$, and the sensor measurement matrix C, which can be retrieved using the methods outlined in [35]. From (1), we note that the problems of estimating $x[k]$ and $x[0]$ are exactly equivalent. Therefore, our focus will be on estimating $x[0]$.

Based on the above ingredients, the problem we consider in this paper is as follows.

Problem 1. *Given the system matrix $A \in \mathbb{R}^{n \times n}$, the sensor measurement matrix $C \in \mathbb{R}^{p \times n}$, the fractional-order coefficients $\{\alpha_i\}_{i=1}^n$ corresponding to each state and the outputs $y_i[k] \in \mathbb{R}^p$ of sensor i, $i \in \{1, \ldots, p\}$, then for every time step $k \in \mathbb{N}$ in (1), we aim to estimate the initial state $x[0] \in \mathbb{R}^n$.*

In what follows, we first provide necessary and sufficient conditions that guarantee the feasibility of the problem, followed by an efficient algorithm to estimate the states.

III. MAIN RESULTS

Prior to going into our main results, we review some essential theory for fractional-order systems, including closed-form expressions for the state dynamics. Using the expansion of the fractional-order derivative in (2), the evolution of the state vector can be written as follows

$$x[k+1] = Ax[k] - \sum_{j=1}^{k+1} D(\alpha, j) x[k+1-j], \qquad (4)$$

where $D(\alpha, j) = \mathrm{diag}(\psi(\alpha_1, j), \psi(\alpha_2, j), \ldots, \psi(\alpha_n, j))$. Alternatively, (4) can be written as

$$x[k+1] = \sum_{j=0}^{k} A_j x[k-j], \qquad (5)$$

where $A_0 = A - D(\alpha, 1)$ and $A_j = -D(\alpha, j+1)$ for $j \ge 1$. Defining matrices G_k as

$$G_k = \begin{cases} I_n & k = 0, \\ \sum_{j=0}^{k-1} A_j G_{k-1-j} & k \ge 1, \end{cases} \qquad (6)$$

we can state the following result.

Lemma 1 ([36]). *The solution to the system described by* (1) *is given by*

$$x[k] = G_k x[0]. \qquad (7)$$

Let $x[0] \in \mathbb{R}^n$ be the initial state of the plant and let $y[0], \ldots, y[k-1] \in \mathbb{R}^p$ be the sensor measurement outputs that are available to the central estimator in k time steps. Using (1) and Lemma 1, we can write

$$y[k] = CG_k x[0] + e[k], \qquad (8)$$

where $e[k]$ represents the artifacts across different measurement channels. If $K \subset \{1, \ldots, p\}$ denotes the set of sensors which are disturbed by artifacts, we can use our assumption that the channels in which the artifacts are present do not change over time to conclude that $\mathrm{supp}(e[k]) \subset K$, where

supp$(\cdot)$ is the *support*, or the number of nonzero elements of a set. After receiving the k sensor outputs $y[0], \ldots, y[k-1]$, the central estimator $\mathcal{C} : (\mathbb{R}^p)^k \to \mathbb{R}^n$ estimates the initial state $x[0]$ of the plant. The estimation is *correct* if $\mathcal{C}(y[0], \ldots, y[k-1]) = x[0]$. More formally, we introduce the following definition.

Definition 1. We say that the central estimator $\mathcal{C} : (\mathbb{R}^p)^k \to \mathbb{R}^n$ can correctly estimate the initial state $x[0]$ in the presence of artifacts in q channels in k time steps if, for any $x[0] \in \mathbb{R}^n$ and *any* sequence of vectors $e[0], \ldots, e[k-1] \in \mathbb{R}^p$ such that $\mathsf{supp}(e[k]) \subset K$, with the cardinality of the set K, $|K| = q$, we have $\mathcal{C}(y[0], \ldots, y[k-1]) = x[0]$ with $y[k] = C G_k x[0] + e[k]$.

Subsequently, the first question one can ask is as follows: What are the required conditions that ensure that the estimation is correct? We provide an answer to this in the next couple of results. First, we consider necessary conditions.

Theorem 1. *Let $k \in \mathbb{N} \setminus \{0\}$ and $E_{q,k}$ be the set of error vectors $(e[0], \ldots, e[k-1]) \in (\mathbb{R}^p)^k$ that satisfy for all $k' \in \{0, \ldots, k-1\}$, $\mathsf{supp}(e[k']) \subset K$ for some $K \in \{1, \ldots, p\}$ with $|K| = q$. Then, the following statements are equivalent:*

(1) *There does not exist a central estimator that can recover the initial state $x[0]$ in k time steps with artifacts across q channels.*
(2) *There exists $x_a, x_b \in \mathbb{R}^n$, $x_a \neq x_b$ and $(e_a[0], \ldots, e_a[k-1]), (e_b[0], \ldots, e_b[k-1]) \in E_{q,k}$ such that $G_k x_a + e_a[k'] = G_k x_b + e_b[k']$ for all $k' \in \{0, \ldots, k-1\}$.*

Proof. The proof of this Theorem readily follows from the distinguishability arguments in the context of observable states, where, given two distinct values x_a, x_b with $x_a \neq x_b$ explaining the data collected with less than q affected channels, it is impossible to unambiguously recover $x[0]$. ∎

We then have a simple necessary and sufficient condition in terms of the matrices C and G_k to characterize the effect of correctly estimating the system states with artifacts being present in q channels.

Theorem 2. *Let $k \in \mathbb{N} \setminus \{0\}$. Then, the following statements are equivalent:*

(1) *The central estimator $\mathcal{C}$ can recover $x[0]$ in k time steps in the presence of artifacts across q channels.*
(2) *For all $z \in \mathbb{R}^n \setminus \{0\}$, $|\mathsf{supp}(CG_0 z) \cup \mathsf{supp}(CG_1 z) \cup \ldots \cup \mathsf{supp}(CG_{k-1} z)| > 2q$.*

Comparisons with LTI systems

Similar results have been obtained by Fawzi et al. [25] for LTI systems. However, there are some crucial differences. Fawzi et al. consider the problem of dealing with errors injected by a malicious agent in the case of a discrete-time linear time-invariant system without inputs. Their version of the above theorem is a generalized criterion for the observability of a linear dynamical system with attacks,

which implies that the initial state $x[0]$ can only be recovered in k time steps if the observability matrix given by $\mathcal{O}_{\mathrm{LTI}} = \begin{bmatrix} C^\mathsf{T} & A^\mathsf{T} C^\mathsf{T} & \ldots & (A^\mathsf{T})^{k-1} C^\mathsf{T} \end{bmatrix}^\mathsf{T}$ has full rank. Furthermore, the maximum number of correctable errors cannot increase beyond $k = n$ measurements, which follows immediately from the Cayley-Hamilton theorem, since, for any z, $k \geq n$, we have $\mathsf{supp}(CA^k z) \subset \mathsf{supp}(Cz) \cup \ldots \cup \mathsf{supp}(CA^{k-1} z)$.

In general, for a linear discrete-time fractional-order system modeled by (1) and the matrices G_k, the necessary and sufficient condition for observability is the existence of a finite time k' (which may be greater than n) such that $\mathrm{rank}(\Xi_{k'}) = n$, where the observability matrix $\Xi_{k'} = \begin{bmatrix} CG_0 & CG_1 & \ldots & CG_{k'-1} \end{bmatrix}^\mathsf{T}$ [36]. Therefore, a remarkable fact is that for a discrete-time fractional-order system, the maximum number of correctable artifacts *can increase* beyond n measurements. This is due to the fact that the terms in the observability matrix for fractional-order systems do not constitute a power series; hence we cannot apply the Cayley-Hamilton theorem as in the linear time-invariant case. Specifically, the terms $G_{k'}$ in the observability matrix $\Xi_{k'}$ are composed of the terms A_j, $j = 0, 1, \ldots, k-1$, which increase with the time step. Put simply, fractional-order dynamics aggregate the effects of *all time* for each new iteration of the state, and hence, this long-term memory becomes relevant in determining the minimum number of time steps in which the system becomes observable. Also, the property of being able to increase the number of correctable artifacts beyond n measurements is ideal from the perspective of our problem, since it assures us that taking more measurements will not go in vain.

Next, we present a result on the number of correctable artifacts as a function of the number of measurements τ.

Theorem 3. *Let $\tau \in \mathbb{N} \setminus \{0\}$ be such that $p\tau \geq k'$, $p \in \mathbb{N} \setminus \{0\}$ where $k' \geq n$ is the minimum index such that $\mathrm{rank}(\Xi_{k'}) = n$. If the initial state $x[0]$ can be estimated in τ time steps in the presence of artifacts in q channels, then*

$$q < \frac{p - \lfloor (k'-1)/\tau \rfloor}{2} \leq \frac{p - k'/\tau + 1}{2}.$$

Remark 1. The problem of actually computing the number of channels affected by artifacts is nontrivial and involves checking that the nullspace of

$$\begin{bmatrix} \mathcal{P}_{K^c} CG_0 \\ \mathcal{P}_{K^c} CG_1 \\ \vdots \\ \mathcal{P}_{K^c} CG_{\tau-1} \end{bmatrix}$$

is not trivial, which, in the worst case, requires computing the rank of 2^p matrices, when $K \subset \{1, \ldots, p\}$. Here, $\mathcal{P}_K$ denotes the projection map onto the components of the set K.

IV. Constructing the Central Estimator

In this section, we will focus on actually constructing the central estimator that can estimate the initial state $x[0]$. Consider the central estimator $\mathcal{C}_0^k : (\mathbb{R}^p)^k \to \mathbb{R}^n$ defined in

a way that $C_0^k(y[0], \ldots, y[k-1])$ is the optimal x solution for the optimization problem

$$\underset{x \in \mathbb{R}^n, K \subset \{1, \ldots, p\}}{\text{minimize}} \quad |K|$$
$$\text{subject to} \quad \text{supp}(y[k'] - CG_{k'}x) \subset K, \quad (9)$$
$$\text{for } k' \in \{0, \ldots, k-1\}.$$

We note that the optimization problem as stated above can have multiple solutions, since the central estimator looks for the smallest set K of channels disturbed by artifacts. In such a case, we consider $\mathcal{C}_0^k(y[0], \ldots, y[k-1])$ to be *any* such solution.

In the next result, we show that the the central estimator $\mathcal{C}_0^k$ is, in some sense, *optimal*.

Theorem 4. *Assume that $x[0]$ can be estimated in k time steps in the presence of q artifacts, or, $|\text{supp}(CG_0z) \cup \ldots \cup \text{supp}(CG_{k-1}z)| > 2q$ for all $z \in \mathbb{R}^n \setminus \{0\}$. Then, C_0^k correctly estimates $x[0]$, that is, for any $x[0] \in \mathbb{R}^n$ and $e[0], \ldots, e[k-1] \in \mathbb{R}^p$ such that $\text{supp}(e[k]) \subset K$ with $|K| \leq q$, we have $\mathcal{C}_0^k(y[0], \ldots, y[k-1]) = x[0]$, where $y[k] = CG_kx[0] + e[k]$.*

The implication of Theorem 4 is that if *any* estimator can estimate $x[0]$ in k time steps in the presence of q artifacts, then $\mathcal{C}_0^k$ also can. However, the optimization problem (9) is NP-hard in general [37].

V. THE ℓ_1 CENTRAL ESTIMATOR

Given the time step $k \in \mathbb{N} \setminus \{0\}$, consider the linear mapping Φ_k defined by

$$\Phi_k : \mathbb{R}^n \to \mathbb{R}^{p \times k}$$
$$x \mapsto \left[\; CG_0x \; | \; CG_1x \; | \; \ldots \; | \; CG_{k-1}x \; \right].$$

Further, define Y_k as the $p \times k$ matrix obtained by stacking $y[0], \ldots, y[k-1]$ as columns, that is,

$$Y_k = \left[\; y[0] \; | \; y[1] \; | \; \ldots \; | \; y[k-1] \; \right] \in \mathbb{R}^{p \times k}.$$

We have already seen that the central estimator $\mathcal{C}_0^k$ computes $x[0]$ from the measurements $y[0], \ldots, y[k-1]$ as the solution of the 'ℓ_0-norm' optimization problem

$$\mathcal{C}_0^k(y[0], \ldots, y[k-1]) = \underset{x \in \mathbb{R}^n}{\text{argmin}} \, \|Y_k - \Phi_k x\|_{\ell_0}. \quad (10)$$

Remark 2. In [38], Candes and Tao show that under certain mild conditions, the 'ℓ_0-norm' can be replaced by the ℓ_1-norm (that is, an ℓ_1-norm relaxation), which leads to a linear program, for which the solutions are more computationally tractable.

From the argument in Remark 2, we have that a central estimator where the 'ℓ_0-norm' is replaced by the ℓ_1-norm can perform the same job without any change in the optimal value of the optimization problem. More generally, following a similar approach to that proposed in [25], given $r \geq 1$, if we measure the magnitude of a row of a matrix by its ℓ_r-norm in $\mathbb{R}^k$, then, the so-called 'ℓ_1/ℓ_r central estimator' can be represented as follows

$$C_{1,r}^k(y[0], \ldots, y[k-1]) = \underset{x \in \mathbb{R}^n}{\text{argmin}} \, \|Y_k - \Phi_k x\|_{\ell_1/\ell_r}, \quad (11)$$

where,

$$\|M\|_{\ell_1/\ell_r} = \sum_{i=1}^{p} \|M_i\|_{\ell_r}. \quad (12)$$

Note that (11) is a convex optimization problem and can be tractably solved.

A. State estimation capability of the ℓ_1/ℓ_r central estimator

Next, we present a result that quantifies the state estimation capability of the ℓ_1/ℓ_r central estimator.

Theorem 5. *The following statements are equivalent:*
(1) *The ℓ_1/ℓ_r central estimator $\mathcal{C}_{1,r}^k$ can correctly estimate $x[0]$ in k time steps in the presence of q artifacts.*
(2) *For all $K \subset \{1, \ldots, p\}$, with $|K| = q$ and for all $\text{III} = \Phi_k z$ with $z \in \mathbb{R}^n \setminus \{0\}$, we have*

$$\sum_{i \in K} \|\text{III}_i\|_{\ell_r} < \sum_{i \in K^c} \|\text{III}_i\|_{\ell_r}. \quad (13)$$

Remark 3. Those who are familiar with the techniques of compressed sensing will immediately notice that the condition $\sum_{i \in K} \|\text{III}_i\|_{\ell_r} < \sum_{i \in K^c} \|\text{III}_i\|_{\ell_r}$ is simply a restatement of the 'nullspace property' that gives necessary and sufficient conditions for the recovery of sparse signals using ℓ_1-norm relaxation [39].

VI. ILLUSTRATIVE EXAMPLES

In this section, we show the performance of the proposed estimator first on a pedagogical toy example, and then on a more realistic system consisting of EEG signals under possible artifacts.

A. Pedagogical Example

First, we consider the performance of the ℓ_1/ℓ_2 central estimator on a system of size $n = p = 4$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.0021 & -0.0273 & -10.4940 & 0.8629 \\ 0 & 0 & 0 & 1 \\ 0.0053 & -0.0682 & -1.7351 & 2.1573 \end{bmatrix},$$

$C = I_{4 \times 4}$, and the fractional-order coefficients $\alpha_1 = 0.10$, $\alpha_2 = 0.15$, $\alpha_3 = 0.60$, and $\alpha_4 = 0.70$. The choice of C being the identity matrix if often suitable in physiological applications, where there are dedicated sensors to capture the evolution of each system state [40]. We assume that there are arbitrary artifacts associated with some measurements of the output y_1, and that the artifacts are ten times the magnitude of the states. Figure 1 shows the comparison between the actual and estimated states which confirms the fact that the system states have been estimated correctly in the presence of artifacts. Note that we have also implicitly verified the statements in Theorems 2 and 3, since, with $p = 4$, $q = 1$, the number of measurements $\tau = 5$, $k' = 4$, and $z = e_4$, where e_4 is the 4-dimensional vector of ones, we have $|\text{supp}(CG_0z) \cup \ldots \cup \text{supp}(CG_4z)| = 4 > 2q = 2$ and $q \leq (p - k'/\tau + 1)/2$. The optimization problems were solved using CVX [41],[42].
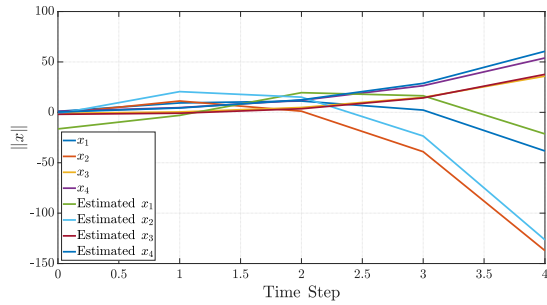
Fig. 1. Actual and estimated system states for the 4×4 toy pedagogical example using the ℓ_1/ℓ_2 central estimator.
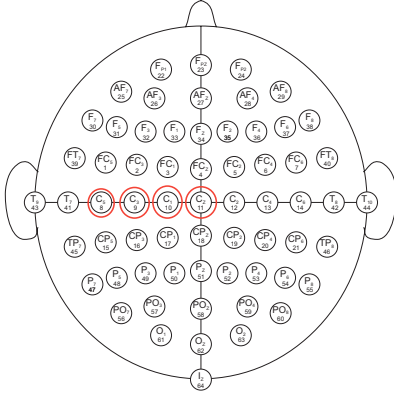

Fig. 2. Sensor distribution for the measurement of EEG. The channel labels are shown along with their corresponding numbers and the selected channels are shown in red.

B. EEG data

We also consider the performance of our algorithm on real neurophysiological signals. In particular, we use 150 measurements taken from 4 channels of a 64-channel EEG signal which records the brain activity of subjects. The distribution of the electrodes and the selected channels, along with the corresponding labels and numbers are shown in Figure 2. The subjects were asked to perform various motor and imagery tasks, and the channels were selected because they are over the motor cortex of the brain, and enable us to predict motor actions such as movement of the hands and feet. The data was collected using the BCI2000 system with a sampling rate of 160 Hz [43], [44]. We assume that there is an 'electrode pop' artifact in the first channel that lasts for 25 milliseconds, following which the electrode picks up noisy data only. The 150 measurements were partitioned into windows of size 6. This was done because the process under consideration is nonlinear, and therefore, the error can become unbounded even under small perturbations, possibly due to the system identification process. The system was identified using the methods described in [35]. As before, we assume that there are dedicated sensors for each system state, that is, $C = I_{4 \times 4}$.

Similar to the pedagogical example, we find that we can invoke the results of the paper, and, in particular, we can estimate the states with artifacts compromising one of the
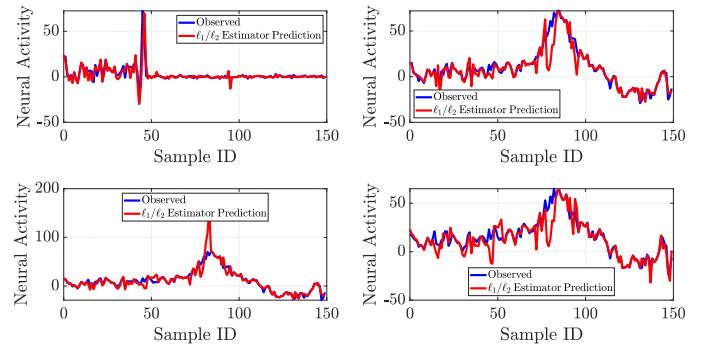

Fig. 3. Performance of the ℓ_1/ℓ_2 central estimator on data collected from a wearable EEG device.

channels. Figure 3 shows the performance of the ℓ_1/ℓ_2 central estimator on the above data. We see that even in the presence of artifacts, we can estimate the system states fairly closely. Therefore, these simulations provide some evidence that the proposed approach might be used in the context of future neuro-wearable device applications.

VII. Conclusion

In this paper, we investigated the problem of estimating the states of a discrete-time fractional-order dynamical system when there are artifacts present in some of the sensor measurements. In particular, we derive necessary and sufficient conditions that enable us to ensure that the estimation of the system states can be done even in the presence of artifacts. We further present some techniques which are inspired by tools used in compressive sensing to estimate the system states. Our results were illustrated on a toy fractional order system as well as more complex systems that represent data collected from a wearable EEG device.

Although FODS have found huge success in modeling the spatiotemporal properties of EEG, some of the properties accounted for by these models actually originate from unknown sources that are external to the system under consideration. Possible future work will be to model these external sources by unknown input stimuli, and then focus on state estimation of the resultant model with inputs. Also, real-time EEG activity can be monitored in order to self-regulate brain function. This is known in the literature as neurofeedback [45], and it would be interesting to study how the introduction of feedback to such a system changes our perspectives on this problem.

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