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Stability and chaos analysis of a novel swarm dynamics with applications to multi-agent systems

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ABSTRACT

This paper presents a novel swarm dynamics and illustrates its applications in automated multi-agent systems. The motion of the particles of the swarm in a particular landscape is governed by an attractant–repellent profile, which has an intimate linkage with the distance separating the particles. Following standard stability and chaos analysis procedures, it is demonstrated that the dynamics indeed simulates a swarm. We adopt a Lyapunov-function based stability and chaos analysis procedure to this effect. The parameterized conditions for which the dynamics exhibits chaotic characteristics are also investigated. Finally, the swarming dynamics is applied to a practical problem, thus elucidating how the proposition can be of use in a real-life situation. Since the dynamics rests on the values of certain parameters, we can control the areas in which we want to use the dynamics by controlling these parameters. The proposed dynamics will be shown to produce convergent, limit cyclic and chaotic behavior. This swarming dynamics can therefore be put to myriad uses depending on the application that is required.

1. Introduction

Swarm behavior, or swarming, can be loosely defined as the collective intelligence exhibited by living or non-living entities, the prime characteristic of which is the *en masse* migration of the individuals in question towards a particular direction. Numerous biological systems exhibit swarming behavior, the most pivotal of them being that exhibited by birds, insects and fish. When applied to inanimate entities, the term may be applied to the control of automated systems of multiple robots (Buhl et al., 2006; Egerstedt and Hu, 2001), programmed vehicles on sea, air or land (Fax and Murray, 2004; Gil et al., 2008a), mechanized tracking (Gil et al., 2008b), rendezvous (Dimarogonas and Kyriakopoulos, 2007), coverage supervision on mobile sensing networks (Cortes et al., 2004), and so on.

The inherently biological phenomenon of swarming (Breder, 1954; Warburton and Lazarus, 1991; Okubo, 1986; Grünbaum and Okubo, 1994; Mogilner and Edelstein-Keshet, 1999; Durrett and Levin, 1994; Gueron and Levin, 1995) has greatly intrigued physicists (Levine and Rappel, 2001; Vicsek et al., 1995; Czirok et al., 1996, 1997; Czirok and Vicsek, 2000; Shimoyama et al., 1996) and mathematicians alike. Computer scientists have also latched onto these patterns by working out efficient optimization algorithms that mimic the real-time behavior of biological swarms. Out of these efforts, algorithms like the ant colony optimization (ACO) algorithm (Bonabeau et al., 1999), the particle swarm optimization (PSO) algorithm (Kennedy et al., 2001; Clerc and Kennedy, 2002) and the bacterial foraging optimization algorithm (BFOA) (Passino, 2002) have evolved.

Since the present work bases itself on the behavior of swarms, a brief review of the various sources and characteristics of swarming behavior present in the currently published literature would not be inappropriate. Extensive amounts of research have been carried out in the field of swarm dynamics, and this has opened up several new and interesting avenues in the aforementioned discipline.

Gazi and Passino (2004) considered an M-individual swarm in an n-dimensional Euclidean space and then modeled the behavior of the constituent particles of the swarm based on the nature of an attractant–repellent profile, or the "σ–profile", as it was referred to by them. Liu and Passino (2004) improved the social foraging swarm profile by modifying it to suit the presence of sensor errors, and even the presence of a considerable amount of various types of noise on the profile, all the while maintaining the robust and cohesive nature for which the profile is known. Leonard and Fiorelli (2001) added support to the above claims by showing allied results based on artificial potentials and virtual leaders for agents with point-mass dynamics. Li (2008) carried out a rigorous study on the stability characteristics of a swarm with general

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directed and weighted topology. Liu et al. (2003) derived conditions for one-dimensional asynchronous swarms to achieve collision-free convergence even in the presence of sensing delays and asynchronism. For the members of a swarm to behave in a manner that justifies the use of the swarming dynamics to solve real-life practical problems, collision avoidance is a critical issue. For example, it is practically useless to propose a dynamics, the constituent particles of which agglomerate after a few iterations. Agglomeration is a very potent issue to be considered, especially when the simplifying assumption of the particles being point masses is not rigorously maintained. Finite particle size does tend to cause problems, the solutions of which are clearly addressed by Liu et al.

Chaos is formally defined as aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions (Strogatz, 1994). The term deterministic means that the system has no random or noisy inputs or parameters. The irregular behavior arises from the system's nonlinearity, rather than from the noisy driving forces. The systematic mathematical study of chaotic systems has hugely developed during the second part of the 20th century (Chen and Dong, 1998; Andrievskii and Fradkov, 2004). Frank et al. (1990) conducted an extensive study of neural network systems, the complex interconnections which help human beings to think. A time-series analysis of chaotic behavior has revealed an intimate linkage of chaos with epileptic seizures. Chen (1988) showed proof of evidence of the presence of low-dimensional strange attractors in several empirical monetary ensembles. Monetary growth is described by a continuous-time deterministic model with delayed feedback. Phase transitions from periodic to chaotic motion are shown.

Chaos analysis in artificial neural networks (ANNs) has occupied significant amounts of space in academia in recent years. Along the lines of Das et al. (2012), we propose a brief survey of the latter. Chen and Aihara (1995) proposed a transiently chaotic neural network (TCNN) as an approximate method for solving combinatorial optimization problems. A transiently chaotic dynamics has been introduced into the neural network. Unlike conventional neural networks with point attractors only, the proposed neural network has richer and more flexible dynamics, so that it can be expected to have higher ability of searching for globally optimal or near-optimal solutions. Nozawa (1992) introduced a neural network model as a globally coupled map. Nozawa's model takes a cue from the network model of Hopfield, possessing a negative self-feedback connection. This model analyzes information in terms of the multitude of maps acting on the constituent nodes of the network, and gives a novel way by which information may be processed. The model is tested on information search using abstruse keywords and the classic traveling salesman problem (TSP). A chaotic approach to solving optimization problems using the technique of simulated annealing has been given by Chen and Aihara (1995). This is different from the model proposed earlier by Aihara et al. in that a negative self-coupling added to the former and thereafter, slow removal of the same produces a transient chaos. This transient chaos is used for combing or self-organizing the search space, thereby producing significantly better results over other techniques which use artificial neural networks (ANNs) to optimize with or without simulated annealing.

It has been found through study that swarms and their behavior can be applied to solve a plethora of engineering problems. Reif and Wang (1999) used a method called the "*social potential fields*" to define inverse-power force laws which dictate "*social relations*" between robots. An individual robot's motion is controlled by the resultant artificial force imposed by other robots and other components of the system. Levine and Rappel (2001) investigated a discrete model consisting of self-propelled particles that obey simple interaction rules. In this work, they demonstrated the self-organization properties of the model and the existence of coherent localized one- and two-dimensional solutions. In the one-dimensional solution, we get a constrained flock which is finitely extant with sharp drops of density down to zero at the edges. Two-dimensional vortex solutions are those in which the particles of interest rotate around a center common to all of them. Random initial conditions, even when confining boundaries are absent, can also engender the latter. Finally, Suzuki and Yamashita (1999) considered a system of multiple mobile robots in which each robot, at infinitely many unpredictable time instants, observes the positions of all the robots and moves to a new position determined by the given algorithm. The work investigates a number of formation problems by robots where they form geometric patterns in the plane where they move. Techniques of converging robots to a given point and moving a system of mobile robots to a point in a given number of finite steps are considered.

In more recent times, Cai et al. (2011) have researched on the problem of swarm stability of high-order linear-time-invariant (LTI) systems with directed graph topology. Necessary and sufficient conditions for swarm stability depending on the graph topology, the dynamics of the agents and the interaction between neighbors are derived. Ranjbar-Sahraei et al. (2012) have proposed a novel decentralized adaptive control scheme for multiagent formation control based on an integration of artificial potential functions with robust control techniques. Robust stability has been demonstrated using Lyapunov-function-based methods, which shows the robustness of the controller with respect to disturbances and system uncertainties. Dolev et al. (2013) have developed a two-phase distributed self-stabilizing scheme for producing a bounded hop-diameter communication graph. Hou and Cheah (2012) have presented a dynamic compound shape control for a swarm of robots. Each basic shape is specified by the corresponding inequality functions. With this new definition, a variety of interesting compound shapes, which are difficult to form by the existing methods, can be easily formed. A Lyapunov-like function is presented for stability analysis of the swarm systems. Guéret et al. (2012) have extended swarm computing to the Semantic Web, a system that promotes the development of the current scenario of the Web by enabling users to find and share information easily. As networking becomes more and more involved and data sets get bigger and bigger, evolutionary and swarm approaches are used to solve these pertinent problems. Yu et al. (2013) have investigated how an inversion of the swarm dynamics can help redefine the rules by which the individual agents operate in order to reach a goal of mutual interest. They have then applied this formulation to the case concerning the point defence of a very important person situated between two swarms, one of which is attacking and the other defending.

Inspired by the flocking dynamics proposed by Cucker and Smale (2007), we present a swarming dynamics and demonstrate its properties sequentially. The dynamics is found to have a conditional stability criterion, ensuring the satisfaction of which we can successfully use the model for multifarious purposes. It turns out to be capable of demonstrating converging, limit cyclic and conditionally chaotic behavior. Since we always want to avoid chaos in the system, proper tuning of the parameters will allow us to work towards this particular goal.

For the demonstration of stability and the arising of conditional chaos, we use the Lyapunov energy function construction method and the nature of the sign of the Lyapunov exponent (Cencini et al., 2010; Wolf et al., 1985). The analytical treatment is supported by a copious number of computer simulations, all of which indicate the validity of the former. Finally, we conclude by proposing a simple yet effective practical problem to which the newly proposed dynamics can be efficiently applied.

2. Proposal of the dynamics

The proposed swarm dynamics consists of M individual agents moving in an n-dimensional Euclidean space. Regardless of their dimensions, the individuals of the swarm are represented by points in space. The motions of the individuals are, however, interlinked. Corresponding to each individual's position, there is a point $x_i \in \Re^n$ and the velocity of the swarm members is dictated by the relation

$$\dot{x}_i = f(x_i - x_j), \tag{1}$$

where $f(x)$ is given by

$$f(x) = \frac{-kx}{[\sigma^2 + \|x\|^2]^\beta}. \tag{2}$$

Eqs. (1) and (2) represent the motion of the swarm members but do not specify how the individuals are motivated towards a particular point of convergence. It represents the interconnection between the individuals as they explore the n-dimensional space. In order to represent realistic biological behavior or multi-agent systems, one requires a function which will guide the swarm towards a region, while they communicate with each other. This can be achieved either by a gradient system or by moving the individuals towards the best point in space by introducing another term in Eq. (1). In the former case the governing equation of motion of the particles looks like

$$\dot{x}_i = -\nabla_{x_i}\xi(x_i) + \sum_{j=1, j \neq i}^{M} \frac{k(x_j - x_i)}{[\sigma^2 + \|x_j - x_i\|^2]^\beta}, \tag{3}$$

where $\xi : \Re^n \longrightarrow \Re$.

The function ξ may be a scalar field representing the concentration of nutrients (as was used in Gazi and Passino's, 2004 paper), or it may represent just a function to be optimized. Eq. (3) will direct the motion of the swarm towards the optima of ξ. Even though the center of the swarm moves towards the optima, the individuals of the swarm may still oscillate about the optima with finite velocities.

Instead of using the gradient system, we may however direct the swarm towards the optima by including a term in Eq. (1) as

$$\dot{x}_i = \sum_{j=1, j \neq i}^{M} \frac{k(x_j - x_i)}{[\sigma^2 + \|x_j - x_i\|^2]^\beta} + \frac{k(p - x_i)}{[\sigma^2 + \|p - x_i\|^2]^\beta}, \tag{4}$$

where $p \in \Re^n$ is the optimum of the artificial potential field considered to simulate the real world application. The field may be the same as ξ. In this case, the individuals communicate to move towards the best possible position achieved. p represents the best possible position detected so far by the swarm and is updated in the course of motion. The best position may represent the point with highest concentration of nutrients; or it may represent the point having maximum temperature in a temperature field; or simply the optima of an objective function. The choice of the field depends on the problem domain. p may also represent the position of a hostile target registered by the detecting system, and missiles are fired to converge at p. We have considered a temperature field in the case of the automated fire engines.

The model we propose in Eq. (4) is able to achieve better result when the scalar field is not well-behaved, i.e. the function contour is not smooth and hence the gradient does not exist at every point in the domain of the field. The first term in the expression for $\dot{x}$ represents the interconnection and communication between the individuals as they traverse. The second term converges the individuals of the swarm towards a particular point in the N-dimensional space. It facilitates modeling swarm behavior where the individuals converge to a particular point and their velocity reduces to zero as they reach the desired point. This is often the situation which we find in nature. A swarm of bees would move in search of nectar and settle down on the flower when they reach there. Here, the point p is analogous to the position of the flower, and the motion of the bees can be modeled by our dynamics. We have demonstrated the application of this modeling to an automated fire engine system in Section 5. The presence of the second term in $\dot{x}_i$ also ensures that even if p varies, the swarm would still be directed towards p. This extends application of the dynamics to multi-agent systems where a target is to be tracked and a swarm is directed towards a time-varying target. Figs. 1–3 show the function $f(x) = -kx/[\sigma^2 + \|x\|^2]^\beta$, which evidently forms the attractant–repellent profile, interlinking the individuals of the swarm. In Gazi and Passino's paper, the attractant–repellent profile did model the foraging swarms. Yet, there remained a discrepancy regarding the attraction tending to infinity at infinite separation. This, however, is not biologically feasible. In biological systems, there is often communication among organisms through release of pheromones. This interaction is possible in a colony or surrounding colonies, but cannot extend to infinite distance. Our dynamics, thus, represents a physical scenario where the individuals in a swarm will lose their co-ordination and cease to attract each other when they are very far apart. They have to be at a critical distance, say eyesight distance or receptive distance, so that attraction can be possible. In this way, the dynamics seems to be in perfect harmony with real

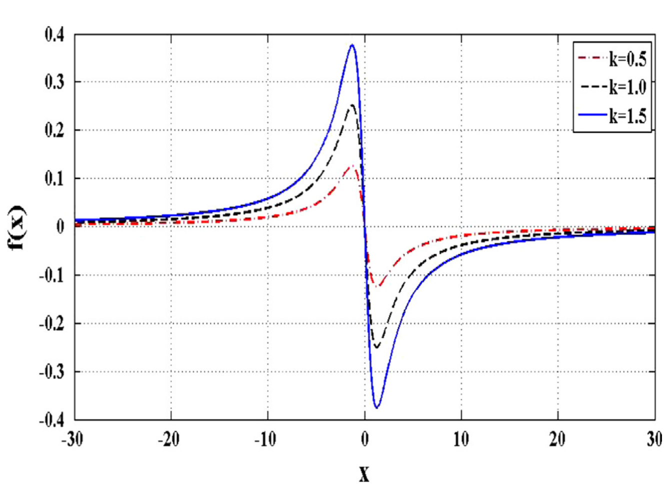

Fig. 1. Effect of varying k on $f(x)$ with $\sigma = 3.5$ and $\beta = 1.2$.

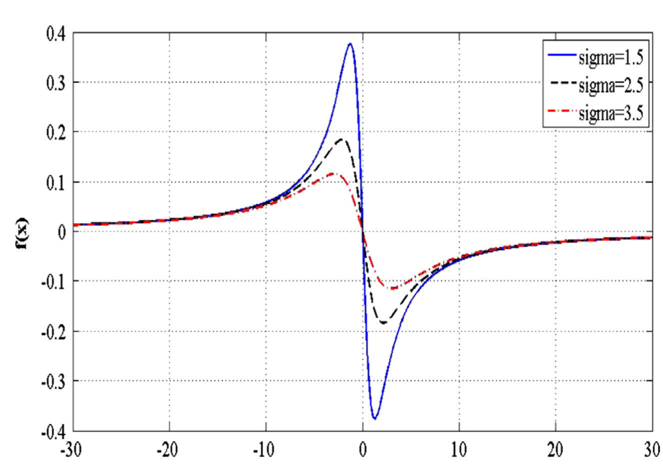

Fig. 2. Effect of varying σ on $f(x)$ with $k = 1.5$ and $\beta = 1.2$.

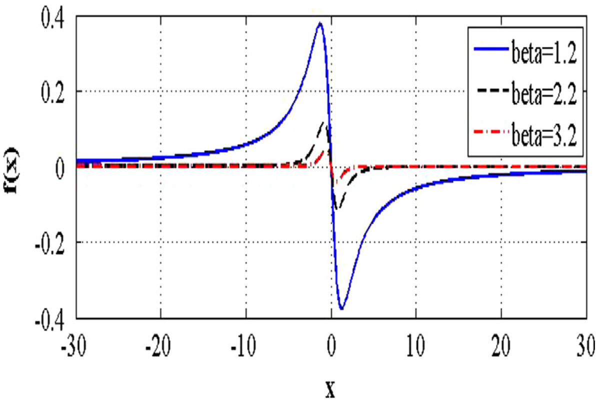

Fig. 3. Effect of varying β on $f(x)$ with $k=1.5$ and $\sigma=3.5$.

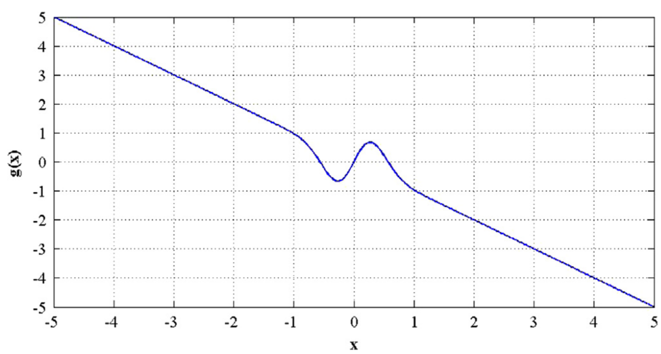

Fig. 4. Attractor–repellent profile in the social foraging swarm of Gazi and Passino (2004).

biological phenomena, since, in nature, the attraction, or desire to move in co-ordination occurs only between individuals in close proximity.

Figs. 1–3 illustrate the attractant–repellent profile in the proposed dynamics and the influence of the variation of different parameters on it. These can be compared with Fig. 4, which shows the attractant–repellent profile in the social foraging swarm dynamics presented by Gazi and Passino (2004). $f(x)$ being an odd function in x ensures that the center of the swarm would move towards the optima, when a gradient system is introduced along with the dynamics. The sensing range of the swarm individuals is governed by the parameters σ, β and k. The function $f(x)$ has an extremum when

$$f'(x)=0 \Rightarrow x=\pm\frac{\sigma}{2\beta-1}, \tag{5}$$

provided that $\beta>\frac{1}{2}$. Upon calculating $f''(x)$ we can show that $f(x)$ has a maxima when $x=-\sigma/(2\beta-1)$ and a minima when $x=+\sigma/(2\beta-1)$ with $\beta>\frac{1}{2}$. Thus, the critical distance up to which attraction is possible depends on σ, β and k, because we can define any particular value of $f(x)$ for which we can consider attraction. As our dynamics models the realistic situation of attraction up to a critical distance due to limitations on sensing, the initial positions of individuals for convergence are determined by the parameters of the dynamics. The assumption that each individual of the swarm knows the position of the other is not difficult to be realized. In multi-agent systems, robots can have a Global Positioning System (GPS) technology built into them. In the case of biological swarms, sensing by sight or chemicals takes place. In addition, the detection of favorable regions in space is achieved by using sensors in multi-agent systems. We desire to model the situation where the individuals of the swarm converge to the desired position and settle there. By varying the parameters of the dynamics, we are also able to model a situation where the individuals reach the desired region, and then exhibit limit cyclic behavior. The situation arises when a swarm of bees reach the favorable region of flowers and move around, collecting nectar from various flowers.

3. Analysis of the conditional stability of the proposed system

Up to this point, we have outlined a dynamics, the working of which is formulated by the fundamental equation

$$\dot{x_i}=\sum_{j=1,j\neq i}^{M}\frac{k(x_j-x_i)}{[\sigma^2+\|x_j-x_i\|^2]^\beta}+\frac{k(p-x_i)}{[\sigma^2+\|p-x_i\|^2]^\beta},$$

where k, σ and β are positive constants. To test the stability of the system we intuitively construct a Lyapunov Energy Function (Malisoff and Mazenc, 2009)

$$L(x_i,x_j)=-\sum_{i=1}^{M}\int_0^{x_i}\left(\sum_{j=1,j\neq i}^{M}\frac{k(x_j-\eta_i)}{[\sigma^2+\|x_j-\eta_i\|^2]^\beta}+\frac{k(p-\eta_i)}{[\sigma^2+\|p-\eta_i\|^2]^\beta}\right)d\eta_i+C, \tag{6}$$

where C is a constant given by

$$C=M^2\left[\int_0^{x_i}\frac{k(x_j-\eta_i)}{[\sigma^2+\|x_j-\eta_i\|^2]^\beta}d\eta_i\right]_{x_i=p,x_j=p}. \tag{7}$$

For $L(x_i,x_j)$ to be a Lyapunov Energy Function, we first take note of the fact that

(a) *Value at critical point(s)*: $L(p,p)=0$.
(b) *Partial derivatives*: The partial derivatives $\partial L/\partial x_i$, $\partial L/\partial x_j$ both exist.
(c) *Value at other points*: Further $L(x_i,x_j)$ will be greater than zero for $x_i,x_j\neq p$ if

$$\sum_{j=1,j\neq i}^{M}\frac{k(x_j-x_i)}{[\sigma^2+\|x_j-x_i\|^2]^\beta}+\frac{k(p-x_i)}{[\sigma^2+\|p-x_i\|^2]^\beta}>0,$$

which after a little bit of algebraic manipulation boils down to

$$x_i<\frac{\sum_{j=1,j\neq i}^{M}\frac{x_j}{[\sigma^2+\|x_j-x_i\|^2]^\beta}+\frac{p}{[\sigma^2+\|p-x_i\|^2]^\beta}}{\sum_{j=1,j\neq i}^{M}\frac{1}{[\sigma^2+\|x_j-x_i\|^2]^\beta}+\frac{1}{[\sigma^2+\|p-x_i\|^2]^\beta}}, \tag{8}$$

when $k=0$.

Now, to test the stability of the system we calculate

$$\frac{dL}{dt}=\frac{\partial L}{\partial x_i}\frac{dx_i}{dt}=-\sum_{i=1}^{M}\left[\sum_{j=1,j\neq i}^{M}\frac{k(x_j-x_i)}{[\sigma^2+\|x_j-x_i\|^2]^\beta}+\frac{k(p-x_i)}{[\sigma^2+\|p-x_i\|^2]^\beta}\right]\frac{dx_i}{dt}=-\sum_{i=1}^{M}\left(\frac{dx_i}{dt}\right)^2<0. \tag{9}$$

Since $dL/dt<0$, the dynamics is asymptotically stable if the condition (8) is met. However, condition (8) signifies a realistic constraint that the system will be stable only when the agents are initially placed keeping their separation within a limit. Thus, this system is asymptotically stable if a certain condition is met and thus can form a swarm.

Figs. 5–8 show the convergent and limit-cyclic behaviors of the system where we choose $\beta=1.2$ and $\sigma=3.5$. These values are chosen for experimental purpose only and accordingly $k(>0)$ is chosen so that the system exhibits stable behavior. Increasing the value of k primarily leads to limit cyclic behavior as shown in these figures and a subsequent increase in k above a threshold,

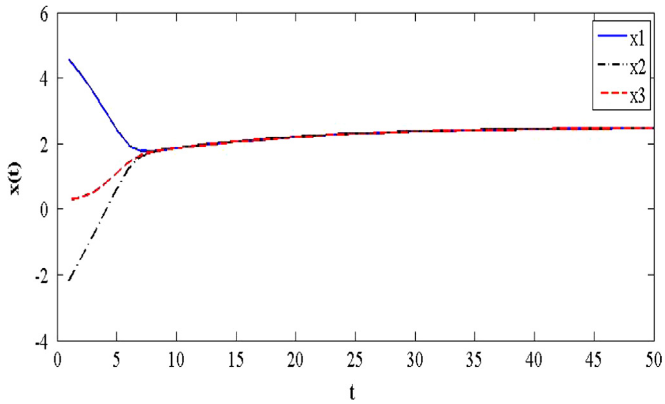

Fig. 5. Stable behavior of $x(t)$ when k=0.5, $\sigma=3.5$, $\beta=1.2$ and p=2.5.

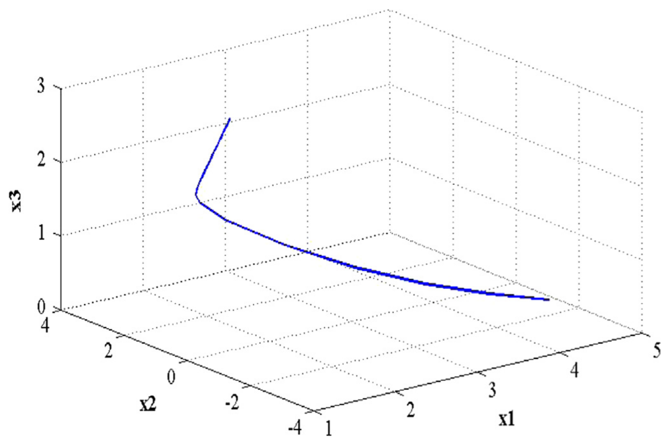

Fig. 6. Phase portrait of the stable behavior of $x(t)$ when k=1.7, $\sigma=3.5$, $\beta=1.2$ and p=2.5.

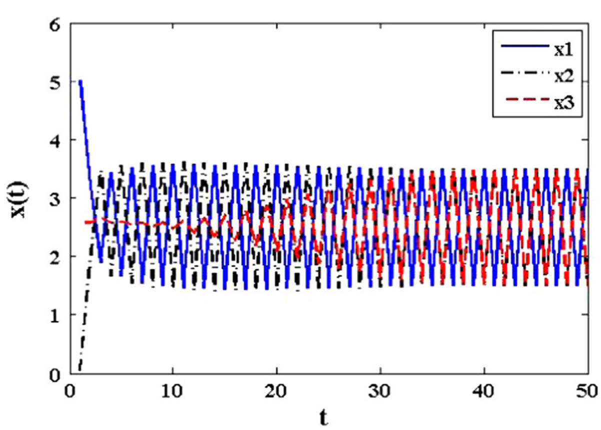

Fig. 7. Limit cyclic behavior of $x(t)$ when k=4.0, $\sigma=3.5$, $\beta=1.2$ and p=2.5.

depending upon the chosen values of σ and β may lead to chaos as we are going to show in the subsequent section.

4. Rise of conditional chaos

In any swarming dynamics, emergence of chaos is a very important situation to be dealt with, as it is known that during chaotic condition the agents of the swarm converge to a region called the strange attractor within which the motion of the agents cannot be predicted from their initial conditions. So the applications where convergence is a prime objective should be devoid of

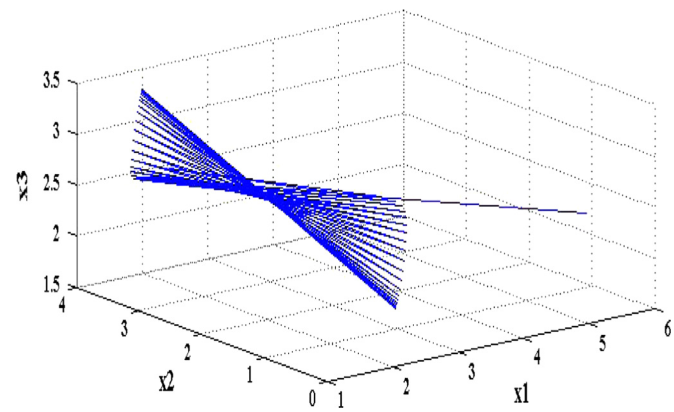

Fig. 8. Phase portrait of the limit cyclic behavior of $x(t)$ when k=4.0, $\sigma=3.5$, $\beta=1.2$ and p=2.5.

chaos and to ensure it we must know the precise value of the parameters for which chaos may arise.

To find out the explicit range of the parameters for which chaos can arise in the proposed system, we will make use of a standard method based on Lyapunov exponents. The Lyapunov exponent (Mosekilde, 1996; Strogatz, 1994) of a dynamic system is given by

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\ln\left|\frac{d(X_{n+1})}{dX_n}\right|,$$

where the system is defined as $\dot{X}=f(X)$, X_n and X_{n+1} being the positions of X calculated at the nth and $(n+1)$th iterations respectively. If $\lambda>0$, the given dynamic system is said to be chaotic. Now, discretizing the differential form of the proposed dynamics, we get

$$\dot{x_i}=\frac{x_i(t+1)-x_i(t)}{(t+1)-t}=x_i(t+1)-x_i(t). \quad (10)$$

Therefore, substitution of (10) into (4) gives

$$x_i(t+1)-x_i(t)=\sum_{j=1,j\neq i}^{M}\frac{k(x_j-x_i)}{[\sigma^2+\|x_j-x_i\|^2]^\beta}+\frac{k(p-x_i)}{[\sigma^2+\|p-x_i\|^2]^\beta}. \quad (11)$$

For the sake of simplicity and convenience, we go for a single dimensional analysis, and replace $\|x_j-x_i\|^2$ as $(x_j-x_i)^2$. Thus, Eq. (11) can be written as

$$x_i^d(t+1)-x_i^d(t)=\sum_{j=1,j\neq i}^{M}\frac{k(x_j^d-x_i^d)}{[\sigma^2+(x_j^d-x_i^d)^2]^\beta}+\frac{k(p^d-x_i^d)}{[\sigma^2+(p^d-x_i^d)^2]^\beta}. \quad (12)$$

Differentiating both sides w.r.t. $x_i^d(t)$

$$\frac{dx_i^d(t+1)}{dx_i^d(t)}=1-k\left[\sum_{j=1,j\neq i}^{M}Z(x_j^d-x_i^d)-Z(x_i^d-p^d)\right], \quad (13)$$

where

$$Z(x)=\frac{1}{[\sigma^2+x^2]^\beta}-\frac{2\beta x^2}{[\sigma^2+x^2]^{\beta+1}}.$$

Now $Z(x)$ has an extremum if

$$\frac{dZ(x)}{dx}=0, \quad (14)$$

which after a little algebraic manipulation gives the roots

$$x=0, \quad \pm\sqrt{\frac{3\sigma^2}{2\beta-1}}. \quad (15)$$

Now, by an elaborate but straightforward calculation which we have not included here due to space insufficiency, it can be shown

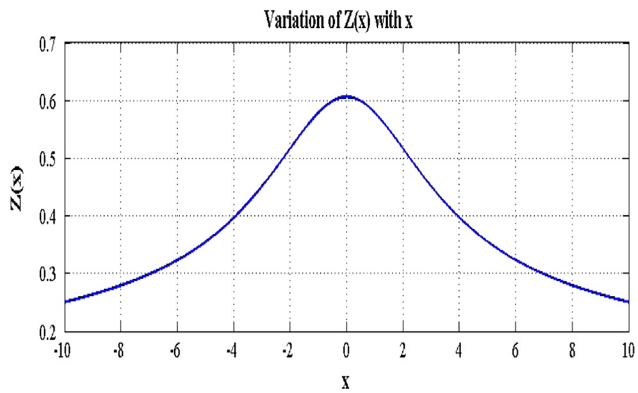

Fig. 9. $Z(x)$ vs x when $\beta \le \frac{1}{2}$.

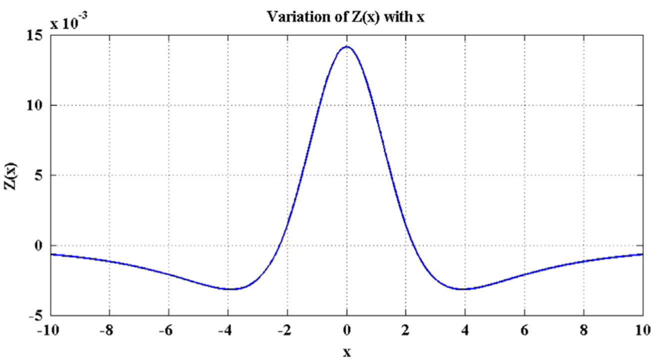

Fig. 10. $Z(x)$ vs x when $\beta > \frac{1}{2}$.

that

$$\left[\frac{d^2 Z(x)}{dx^2}\right]_{x=0} < 0.$$

So $Z(x)$ has a maxima at $x=0$ and

$$Z_{max} = \frac{1}{\sigma^{2\beta}}.$$

From Eq. (15), we see that if $\beta \le \frac{1}{2}$, $x=0$ is the only extremum of $Z(x)$ which is a maxima and therefore $Z(x)$ does not have any distinct minima. But we see that when $x<0$, $Z'(x)>0$, i.e., $Z(x)$ is increasing and again when $x>0$, $Z'(x)<0$, i.e., $Z(x)$ is decreasing. Moreover $x \to \pm\infty \Rightarrow Z(x) \to 0$. Thus, we can conclude that the x-axis is an asymptote of $Z(x)$, and the lower bound of $Z(x)$ can be taken as zero. So for $\beta \le \frac{1}{2}$

$$Z_{min} = 0.$$

Fig. 9 depicts the plot of $Z(x)$ vs x when $\beta \le \frac{1}{2}$ and this supports our inference. When $\beta > \frac{1}{2}$, it can be shown that $Z = Z_{min}$ when

$$x = \pm\sqrt{\frac{3\sigma^2}{2\beta - 1}}$$

and

$$Z_{min} = \frac{-2\sigma^2}{\left[\sigma^2\left(\frac{3}{2\beta-1}+1\right)\right]^{\beta+1}}.$$

as depicted in Fig. 10. So, the function $Z(x)$ is bounded. Now, to ensure stability

$$\left|\frac{dx_i^d(t+1)}{dx_i^d(t)}\right| \le 1$$

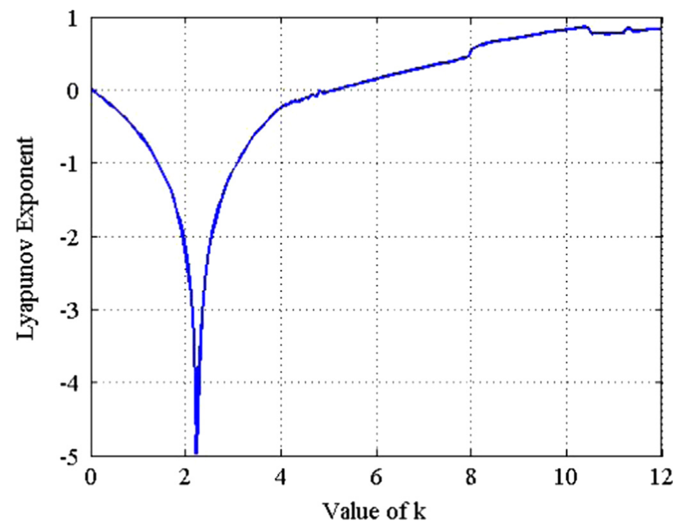

Fig. 11. Variation of Lyapunov exponent with the positive values of k when $\sigma=3.5$, $\beta=1.2$ and $p=2.5$.

$$\Rightarrow -1 \le 1 - k\left[\sum_{j=1, j\ne i}^{M} Z(x_j^d - x_i^d) - Z(x_i^d - p^d)\right] \le 1.$$

But as Z is bounded and $k>0$

$$1 - kMZ_{max} \le \frac{dx_i^d(t+1)}{dx_i^d(t)} \le 1 - kMZ_{min}. \quad (16)$$

Therefore

$$1 - kMZ_{max} \ge -1 \Rightarrow k \le \frac{2}{MZ_{max}}, \quad (17)$$

and

$$1 - kMZ_{min} \le 1. \quad (18)$$

Combining Eqs. (17) and (18), we get

$$0 \le k \le \frac{2}{MZ_{max}}. \quad (19)$$

For emergence of chaos

$$\left|\frac{dx_i^d(t+1)}{dx_i^d(t)}\right| > 1.$$

So from (16) it can be concluded that

$$1 - kMZ_{min} > 1 \Rightarrow kMZ_{min} < 0. \quad (20)$$

Now it is possible only if $Z_{min} < 0$ as $k>0$. Again we need

$$1 - kMZ_{max} < -1 \Rightarrow k > \frac{2}{MZ_{max}}. \quad (21)$$

Combining (20) and (21) we get that the system will show chaotic behavior iff $Z_{min}<0$ and $k > 2/MZ_{max}$ which in turn gives

$$\beta > \tfrac{1}{2},$$

and

$$k > \frac{2}{MZ_{max}}.$$

The chaotic behavior of the dynamics is duly illustrated in Fig. 12 where the both the phase plot and the $x(t)$ vs t plot are given. A plot of the Lyapunov exponent vs value of k is also given in Fig. 11. Here β is chosen to be 1.2 as $\beta > 0.5$ condition is necessary to make Lyapunov exponent positive for $k > 2/MZ_{max}$ ensuring chaos. σ is chosen as 3.5 and accordingly in Fig. 12 k is chosen as 11 so that it satisfies condition (21).

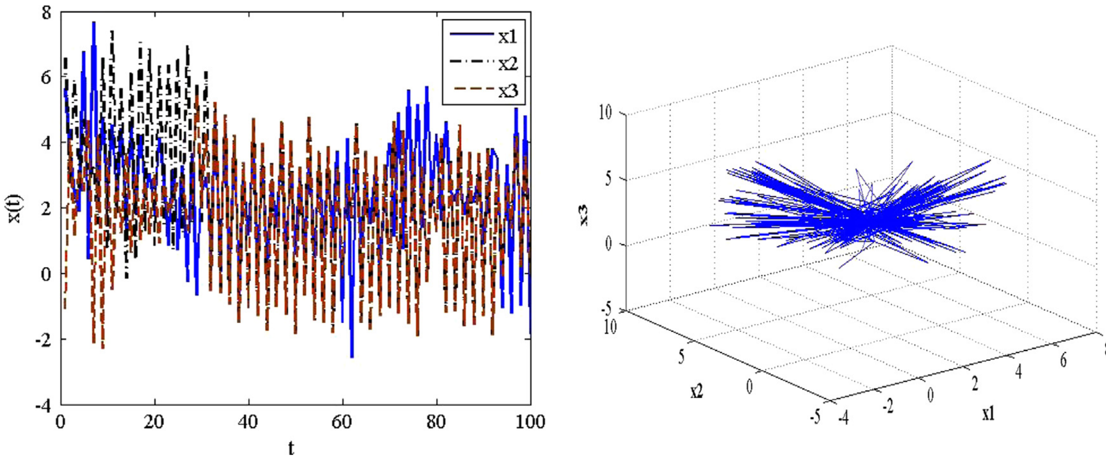

Fig. 12. Chaotic behavior of $x(t)$ when $k=11.0$, $\sigma=3.5$, $\beta=1.2$ and $p=2.5$; left figure: plot of $x(t)$ vs t, right figure: phase portrait.

5. Implementation in a real-life problem

The dynamics we present in this paper can be purposefully incorporated in multi-agent systems and those systems can effectively be used in real-life problems. This section illustrates a simple but effective use of the swarming dynamics and its machine-simulations.

The stable behavior of the dynamics described has been successfully applied to a multi-robot system where each robot is an automatic fire-extinguisher. Now every robot in this system has a temperature sensor having the ability to sense the local temperature and can communicate with others through some wireless link making them able to send and receive the temperature-related data. Let us assume that a fire has started somewhere in the region where the extinguishers are deployed. So there will be a spatially varying temperature field in that region with its maxima at the point of the fire. Now, if we apply the proposed dynamics in the motion of those fire-tender robots, they can automatically detect the place where the fire has broken out and extinguish it. The robot must sense the temperature field and the system should converge to the point where the fire started. We have shown that the dynamics that we have proposed converges to the point p for certain values of k, σ and β ensuring that there will be no chaotic fluctuations. But in this problem, the required point of convergence is not known beforehand and is thus to be automatically detected by the agents. This can be achieved by the inclusion of a simple memory system in which each agent can keep track of the best position i.e., the highest temperature found by them till now. This best position will be designated as p, and thus the swarm will have a tendency to move towards this position. However, as there remains attraction and repulsion among the agents, they also have a tendency to explore new areas and there will be a strong probability that p will be updated in their course of motion. Thus ultimately the whole swarm will be able to converge to the position of the fire provided that the positions of the agents are initiated keeping the diversity. However, the dynamics does not guarantee that all the robots will converge to the position of fire, but can assure that they will converge to the best position ever achieved.

Here we give some instances of computer-simulation of this real life problem where the temperature field is assumed to be a shifted inverted sphere function with its maxima at different points for different examples. Figs. 13 and 14 show that the convergence of all the agents to the respective maxima is achieved in every case. However for a wide search-space, a higher number of agents yield better result and the value of σ should be accordingly high to get rid of chaos.

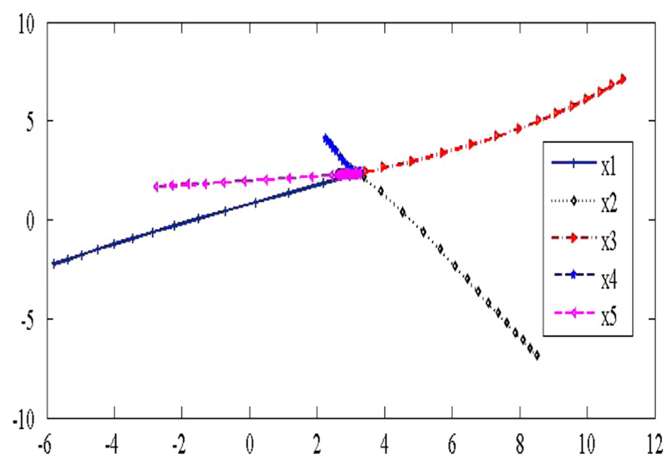

Fig. 13. Convergence of the agents in a 2D field when $M=10$, $\sigma=3.5$, $k=1.7$, $\beta=1.2$ and the maxima is at $(2.5, 2.5)$.

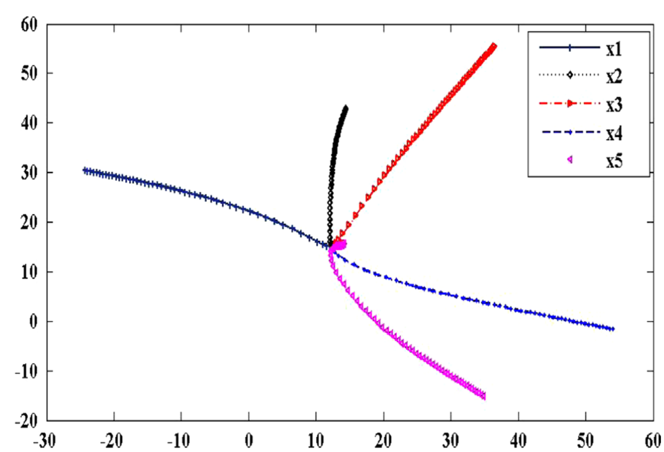

Fig. 14. Convergence of the agents in a 2D field when $M=100$, $\sigma=10.5$, $k=1.7$, $\beta=1.2$ and the maxima is at $(15, 15)$.

Fig. 13 shows the motion of five agents where the total number of agents $M=10$, $\sigma=3.5$, $k=1.7$. Fig. 14 shows another case where these values are 100, 40.5 and 1.7 respectively. The maxima of the temperature field was at $(2.5, 2.5)$ and $(15, 15)$ in these two cases.

From the successive positions of the agents it is clear that in the first case the search space is smaller and the speed of the convergence is more than that in the second case.

The situation is indeed averse when the fire spreads from the starting point. This can often take place when there is the presence

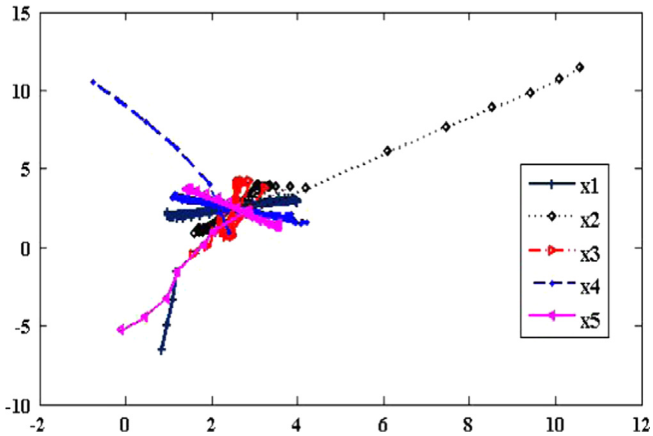

Fig. 15. Convergence of the agents in a 2D field when $M=10$, $\sigma=3.5$, $k=6.0$, $\beta=1.2$ and the maxima is at $(2.5, 2.5)$.

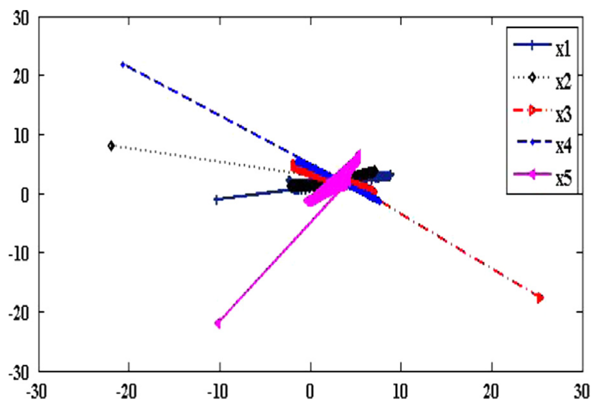

Fig. 16. Convergence of the agents in a 2D field when $M=100$, $\sigma=40.5$, $k=130$, $\beta=1.2$ and the maxima is at $(2.5, 2.5)$.

of inflammable articles in the vicinity of the origin of fire. In such a situation, it is required that the automated fire engines sweep the neighborhood region, and not just converge at the starting point of fire. The dynamics which we have presented is able to perform this operation by exploiting the limit cyclic behavior. The extensive analysis of the dynamics provided us the parameter tuning requirements that would cause the agents to surround the point of fire. The flames, even after spreading from the starting point, would be efficiently doused. The essence of the chaos, limit cycle and convergence analysis lies in the ability to subtly adjust the dynamics of the agents based on the domain necessity. We can also organize the agents such that half of the agents are parameterized to undergo limit cyclic behavior and the remaining agents converge to the best obtained maximum temperature point. This would create a dual effect of extinguishing the flames at the origin of fire, as well as prevent any possibility of spreading.

Figs. 15 and 16 demonstrate the performance of the fire-extinguishing system that exploits limit cyclic behavior and thus the agents explore the neighborhood area of the temperature maxima. Fig. 15 shows the limit-cyclic oscillations of five agents when $M=10$, $\sigma=3.5$, $k=6.0$. In Fig. 16, however, $M=100$, $\sigma=40.5$, $k=130$. In this case a greater number of agents need greater values of σ and k to achieve limit-cyclic behavior. In both the figures, it can be seen that the agents effectively traverse the area surrounding the maxima of the temperature contour.

The dynamics adapts itself suitably to various problem domains. Even though we have simulated the multi-agent fire engines, the dynamics could also be used in a patrolling system. The situation is much akin the one in which the fire spreading is prevented by surrounding the fire engines around the starting point. The limit cyclic behavior finds utmost prominence in such applications. Situations in which we do want the agents to converge can be achieved as in the case of the engines converging to the point of fire. Thus, the fire engines simulation is just a specialized case of the variety of real life problems of target detection, patrolling, optimization that the dynamics can be utilized in. Only requirement is the requisite parameter tuning for the relevant field. A similarity in all the applications is the presence of a destination and a multi-agent system. For a social foraging, destination is the food source, the organisms constitute the multi-agent system; for patrolling, the military base may be the destination, and helicopters are the agents. Proceeding by a similar analogy, the target and the missiles in a radar, the origin of fire and the fire engines are the destination and multi-agent system pairs for the two situations. An optimization problem may consider the optima of the objective function as the destination, and the swarm traverses the search space to find the optima.

6. Conclusion

The ideas presented in this paper have taken concepts from the real-time behavior of multi-agent swarms. We have successfully demonstrated that the system is conditionally stable and that the dynamics can be applied at ease to all those cases which ensure the satisfaction of the stability condition. The attractant–repellent profile of the proposed dynamics has been compared to that of the social foraging swarm dynamics. It has indeed been found that in terms of practicality, the attractant–repellent profile of the proposed dynamics is conceptually superior to the social foraging swarm dynamics in the sense that it prevents infinite attraction between any two members of the swarm at infinite distance. The analytical treatments undertaken indicate that this system can indeed be stable and hence is able to model a swarm. Accompanying this observation is the fact that the dynamics is able to exhibit convergent, limit cyclic and chaotic behavior. It has also been seen that the inclusion of the second term in the equation that defines the preliminary behavior of the dynamics tends to converge the particles of the swarm towards a particular point. This highly significant observation has been used in the simulation of an automatic system of fire detection and extinguishing by a system of fire engines. In addition, we can also judiciously use the limit cyclic behavior of the system if we keep in mind the fact that when a fire does break out in a particular area, it is never localized.

The possible future extension of this work can be the case of time-varying field which is more realistic in the case of a moving "target" or "threat". Again, the function $f(x_i - x_j)$ in Section 2 can itself be dynamic such that $f = f(x,t)$ and that will be more analogous to biological swarm where the velocity updation rule of each agent may change with time.

Appendix A. Evaluation of the constant term in the Lyapunov energy function

In Section 3, during the construction of the Lyapunov function for proving stability criteria, we have incorporated a constant C in order to make $L(p,p) = 0$ in Eq. (6). The constant C has been given as

$$C = M^2 \left[\int_0^{x_i} \frac{k(x_j - \eta_i)}{[\sigma^2 + \| x_j - \eta_i \|^2]^\beta} d\eta_i \right]_{x_i = p, x_j = p}.$$

Now to make $L(p,p)=0$, we need

$$\left[-\sum_{i=1}^{M}\int_0^{x_i}\left(\sum_{j=1, j\neq i}^{M}\frac{k(x_j-\eta_i)}{[\sigma^2+\|x_j-\eta_i\|^2]^\beta}+\frac{k(p-\eta_i)}{[\sigma^2+\|p-\eta_i\|^2]^\beta}\right)d\eta_i\right]_{x_i=p, x_j=p}+C=0. \quad (22)$$

Now, when $x_i=p$ and $x_j=p\ \forall\ i,j\in Z^+$, the L.H.S of Eq. (22) will readily boil down to

$$-M^2\left[\int_0^{x_i}\frac{k(x_j-\eta_i)}{[\sigma^2+\|x_j-\eta_i\|^2]^\beta}d\eta_i\right]_{x_i=p, x_j=p}, \quad (23)$$

which is exactly the same as the expression of C given in Eq. (7) with only the sign reversed. Obviously, when added with the constant C, expression (23) results to zero. So, from Eq. (22), it is clear that $L(p,p)$ is zero when $x_i=p$ and $x_j=p\,\forall i,j\in Z^+$.

Now we will evaluate this constant C for the special case where $x_i, x_j\in\Re$, i.e. $n=1$. In this condition we can replace $\|x_j-x_i\|^2$ by $(x_j-x_i)^2$. So the definite integral in the expression (23) becomes

$$\int_0^{x_i}\frac{k(x_j-\eta_i)}{[\sigma^2+(x_j-\eta_i)^2]^\beta}d\eta_i. \quad (24)$$

Now substituting $\sigma^2+(x_j-\eta_i)^2$ by z and following the steps of standard integral calculation the integral (24) results in

$$\frac{k}{2(\beta-1)}\left[\frac{1}{[\sigma^2+(x_j-x_i)^2]^{\beta-1}}-\frac{1}{(\sigma^2+x_j^2)^{\beta-1}}\right],$$

which in turn gives

$$\frac{k}{2(\beta-1)}\left[\frac{1}{\sigma^{2(\beta-1)}}-\frac{1}{(\sigma^2+p^2)^{\beta-1}}\right],$$

when $x_i=p$ and $x_j=p$.

So the value of the integral in the expression (24) is ultimately given by

$$\frac{-kM^2}{2(\beta-1)}\left[\frac{1}{\sigma^{2(\beta-1)}}-\frac{1}{(\sigma^2+p^2)^{\beta-1}}\right];$$

and hence C is given by

$$C=\frac{kM^2}{2(\beta-1)}\left[\frac{1}{\sigma^{2(\beta-1)}}-\frac{1}{(\sigma^2+p^2)^{\beta-1}}\right], \quad (25)$$

when $x_i, x_j\in\Re$.

Appendix B. Symbols used in the paper

Symbol (s)	Definition
$x_i(t)$	position of the ith particle at tth instant
$x_i^d(t)$	dth dimension of $x_i(t)$
k,σ,β	parameters that control the velocity of the agents according to Eqs. (1) and (2)
$\xi(x_i)$	a scalar function of x_i; $\xi:\Re^n\to\Re$
$L(x_i,x_j)$	Lyapunov energy function
M	total number of agents
p	convergence point of the system (stable limit point), modeled as the optima of an artificial scalar field
λ	Lyapunov exponent
Z	a function of x defined in Eq. (13)

References

Andrievskii, B.R., Fradkov, A.L., 2004. Control of chaos: methods and applications. Autom. Remote Control 64 (April (5)), 673–713.
Bonabeau, E., Dorigo, M., Theraulaz, G., 1999. Swarm Intelligence: From Natural to Artificial Systems. Oxford University Press, New York
Breder, C.M., 1954. Equations descriptive of fish schools and other animal aggregations. Ecology 35 (3), 361–370.
Buhl, J., Sumpter, D.J.T., Couzin, I.D., Hale, J.J., Despland, E., Miller, E.R., Simpson, S.J., 2006. From disorder to order in marching locusts. Science 312 (June (5778)), 1402–1406.
Cai, N., Xi, J.-X., Zhong, Y.-S., 2011. Swarm stability of high-order linear time-invariant swarm systems. IET Control Theory Appl. 5 (2), 402–408.
Cencini, M., Cecconi, F., Vulpiani, A., 2010. Chaos From Simple Models to Complex Systems. World Scientific Publishing Co.
Chen, P., 1988. Empirical, theoretical evidence of economic chaos. Syst. Dyn. Rev. 4, 81.
Chen, L.N., Aihara, K., 1995. Chaotic simulated annealing by a neural network model with transient chaos. Neural Netw. 8 (6), 915–930.
Chen, G., Dong, X., 1998. From Chaos to Order: Methodologies, Perspectives and Applications. World Scientific, Singapore
Clerc, M., Kennedy, J., 2002. The Particle Swarm-Explosion, Stability, and Convergence in a Multidimensional Complex Space. IEEE Trans. Evol. Comput. 6 (February (1)), 58–73.
Cortes, J., Martinez, S., Karatas, T., Bullo, F., 2004. Coverage control for mobile sensing networks. IEEE Trans. Robot. Autom. 20 (April (2)), 243–255.
Cucker, F., Smale, S., 2007. Emergent behavior in flocks. IEEE Trans. Autom. Control 52 (May (5)), 852–862.
Czirok, A., Vicsek, T., 2000. Collective behavior of interacting self-propelled particles. Physica A 281, 17–29.
Czirok, A., Ben-Jacob, E., Cohen, I., Vicsek, T., 1996. Formation of complex bacterial colonies via self-generated vortices. Phys. Rev. E 54 (August (2)), 1791–1801.
Czirok, A., Stanley, H.E., Vicsek, T., 1997. Spontaneously ordered motion of self-propelled particles. J. Phys. A: Math Nucl. Gen. 30, 1375–1385.
Das, S., Halder, U., Maity, D., 2012. Chaotic dynamics in social foraging swarms—an analysis. IEEE Trans. Syst. Man. Cybern.: Cybern. 42 (August (4)), 1288–1293.
Dimarogonas, D.V., Kyriakopoulos, K.J., 2007. On the rendezvous problem for multiple nonholonomic agents. IEEE Trans. Autom. Control 52 (May (5)), 916–922.
Dolev, S., Segal, M., Shpungin, H., 2013. Bounded-Hop energy-efficient liveness of flocking swarms. IEEE Trans. Mob. Comput. 12 (March (3)), 516–528.
Durrett, R., Levin, S., 1994. The importance of being discrete (and spatial). Theor. Popul. Biol. 46, 363–394.
Egerstedt, M., Hu, X., 2001. Formation constrained multi-agent control. IEEE Trans. Robot. Autom. 17 (December), 947–951.
Fax, J.A., Murray, R.M., 2004. Information flow and cooperative control of vehicle formations. IEEE Trans. Autom. Control 49 (September (9)), 1465–1476.
Frank, G.W., Lookman, T., Nerenberg, M.A.H., Essex, C., Lemieux, J., Blume, W., 1990. Chaotic time series analysis of epileptic seizures. Physica D 46, 427.
Gazi, V., Passino, K.M., 2004. Stability analysis of social foraging swarms. IEEE Trans. Syst. Man Cybern. B 34 (February (1)), 539–557.
Gil, A., Passino, K.M., Ganapathy, S., Sparks, A., 2008a. Cooperative task scheduling for networked uninhabited air vehicles. IEEE Trans. Aerosp. Electron. Syst. 44 (April (2)), 561–581.
Gil, A., Passino, K.M., Cruz, J.B., 2008b. Stable cooperative surveillance with information flow constraints. IEEE Trans. Control Syst. Technol. 16 (September (5)), 856–868.
Grünbaum, D., Okubo, A., 1994. Modeling social animal aggregations. In: Frontiers in Theoretical Biology, vol. 100. Springer-Verlag, New York, pp. 296–325.
Guéret, C., Schlobach, S., Dentler, K., Schut, M., Eiben, G., May 2012. Evolutionary and swarm computing for the semantic web. IEEE Comput. Intell. Mag., 16–31.
Gueron, S., Levin, S.A., 1995. The dynamics of group formation. Math. Biosci. 128, 243–264.
Hou, S.P., Cheah, C.C., 2012. Dynamic compound shape control of robot swarm. IET Control Theory Appl. 6 (3), 454–460.
Kennedy, J., Eberhart, R.C., Shi, Y., 2001. Swarm Intelligence. Morgan Kaufmann, San Mateo, CA
Leonard, N.E., Fiorelli, E., December 2001. Virtual leaders, artificial potentials and coordinated control of groups. In: Proceedings of Conference Decision Control, Orlando, FL, pp. 2968–2973.
Levine, H., Rappel, W.-J., 2001. Self-organization in systems of self-propelled particles. Phys. Rev. E 63 (January (1)), 017 101-1–017 101-4.
Li, W., 2008. Stability analysis of swarms with a general topology. IEEE Trans. Syst. Man Cybern. B 38 (August (4)), 1084–1097.
Liu, Y., Passino, K.M., 2004. Stable social foraging swarms in a noisy environment. IEEE Trans. Autom. Control 49 (January (1)), 30–44.
Liu, Y., Passino, K.M., Polycarpou, M.M., 2003. Stability analysis of one-dimensional asynchronous swarms. IEEE Trans. Autom. Control 48 (October (10)), 1848–1854.
Malisoff, M., Mazenc, F., 2009. Construction of Strict Lyapunov Functions. Springer.
Mogilner, A., Edelstein-Keshet, L., 1999. A nonlocal model for a swarm. J. Math. Biol. 38, 534–570.
Mosekilde, E., 1996. Topics in Nonlinear Dynamics. World Science, London
Nozawa, H., 1992. A neural-network model as a globally coupled map and applications based on chaos. Chaos 2 (3), 377–386.

Okubo, A., 1986. Dynamical aspects of animal grouping: swarms, schools, flocks, and herds. Adv. Biophys. 22, 1–94.
Passino, K.M., 2002. Biomimicry of bacterial foraging for distributed optimization and control. IEEE Control Syst. Mag. 22 (June (3)), 52–67.
Ranjbar-Sahraei, B., Shabaninia, F., Nemati, A., Stan, S.D., 2012. A novel robust decentralized adaptive fuzzy control for swarm formation of multiagent systems. IEEE Trans. Ind. Electron. 59 (August (8)), 3124–3134.
Reif, J.H., Wang, H., 1999. Social potential fields: a distributed behavioral control for autonomous robots. Robot. Auton. Syst. 27, 171–194.
Shimoyama, N., Sugawa, K., Mizuguchi, T., Hayakawa, Y., Sano, M., 1996. Collective motion in a system of motile elements. Phys. Rev. Lett. 76 (May (20)), 3870–3873.
Strogatz, S.H., 1994. Nonlinear Dynamics and Chaos. Perseus Books, MA
Suzuki, I., Yamashita, M., 1999. Distributed anonymous mobile robots: formation of geometric patterns. SIAM J. Comput. 28 (4), 1347–1363.
Vicsek, T., Czirok, A., Ben-Jacob, E., Cohen, I., Shochet, O., 1995. Novel type of phase transition in a system of self-driven particles. Phys. Rev. Lett. 75 (August (6)), 1226–1229.
Warburton, K., Lazarus, J., 1991. Tendency-distance models of social cohesion in animal groups. J. Theor. Biol. 150, 473–488.
Wolf, A., Swift, J.B., Swinney, H.L., Vastano, J.A., 1985. Determining Lyapunov exponents from a time series. Physica D 16.
Yu, A.R., Thompson, B.B., Marks , R.J., 2013. Competitive evolution of tactical multiswarm dynamics. IEEE Trans. Syst. Man Cybern.: Syst. 43 (May (3)), 563–569.